Math 180 Written Homework

Assignment #5

Due **Tuesday, October 7th** at the beginning of your discussion class.

Directions. You are welcome to work on the following problems with other MATH 180 students, but your solutions must be hand-written, by your own hand, representing your understanding of the material. Word-by-word copying from another student or any other source is unacceptable. Any work without the proper justification will receive no credit. The list of problem solutions is to be submitted to your TA at the beginning of the discussion class listed above. No late homework will be accepted.

1. Calculate the following derivatives. Do not simplify your answers.

   (a) $\sqrt{x^2 - 7x}$

   (b) $e^{x \cos x}$

   (c) $\sin(\cos x)$

   (d) $\cos^3 x$

   (e) $x^2 e^{-5x}$

   (f) $\left(\dfrac{t^3+1}{t^3-1}\right)^4$

   (g) $\tan^2 x + \tan(x^2)$

   (h) $\dfrac{8}{\sqrt{4+3x}}$

   (i) $e^{\sqrt{x^2-1}+\frac{1}{x}}$

   SOLUTION:

   (a) $\dfrac{1}{2}(x^2-7x)^{-1/2} \cdot (2x-7)$

   (b) $e^{x\cos x} \cdot (\cos x - x \sin x)$

   (c) $\cos(\cos x) \cdot (-\sin x)$

   (d) $3\cos^2 x \cdot (-\sin x)$

   (e) $2x \cdot e^{-5x} + x^2 \cdot e^{-5x} \cdot -5$

   (f) $4\left(\dfrac{t^3+1}{t^3-1}\right)^3 \cdot \dfrac{(t^3-1)(3t^2)-(t^3+1)(3t^2)}{(t^3-1)^2}$

   (g) $2\tan x \cdot \sec^2 x + \sec^2(x^2) \cdot 2x$

   (h) First rewrite the quotient as $8(4+3x)^{-1/2}$. Then the derivative is $-4(4+3x)^{-3/2} \cdot 3$

   (i) $e^{\sqrt{x^2-1}+\frac{1}{x}} \cdot \left(\dfrac{1}{2}(x^2-1)^{-1/2} \cdot 2x - x^{-2}\right)$

2. A differentiable function f satisfies $f(3) = 5$, $f(9) = 7$, $f'(3) = 11$, and $f'(9) = 13$. Find an equation for the tangent line to the curve $y = f(x^2)$ at the point $(x, y) = (3, 7)$.

SOLUTION: The derivative $\frac{dy}{dx}$ is computed using the Chain Rule.

$$\frac{dy}{dx} = f'(x^2) \cdot \frac{d}{dx}x^2 = f'(x^2) \cdot 2x$$

At $x = 3$ we have

$$\left.\frac{dy}{dx}\right|_{x=3} = f'(3^2) \cdot 2(3) = f'(9) \cdot 6 = 13 \cdot 6 = 78.$$

This is the slope of the line tangent to $y = f(x^2)$ at the point $(3, 7)$. Thus, an equation for the tangent line in point-slope form is

$$y - 7 = 78(x - 3).$$

3. (a) Verify that point $P(0, 1)$ lies on the curve $\sqrt{x^2 + y^2} = \tan(\frac{\pi}{4}(x + y))$.

   (b) Find the equation of the line tangent to the curve at P.

SOLUTION: (a)

$$\begin{aligned} \sqrt{0^2 + 1^2} &= \tan\left(\frac{\pi}{4}(0 + 1)\right) \\ 1 &= \tan\left(\frac{\pi}{4}\right) = 1 \end{aligned}$$

so the point is on the curve.

(b)

$$\frac{1}{2}(x^2 + y^2)^{-1/2} \cdot (2x + 2y\frac{dy}{dx}) = \sec^2\left(\frac{\pi}{4}(x + y)\right) \cdot \left(\frac{\pi}{4}(1 + \frac{dy}{dx})\right)$$

Now plug in $(0,1)$ and solve for $\dfrac{dy}{dx}$:

$$\begin{aligned}
\frac{1}{2}(0^2+1^2)^{-1/2}\cdot(2\cdot 0+2\frac{dy}{dx}) &= \sec^2\left(\frac{\pi}{4}(0+1)\right)\cdot\left(\frac{\pi}{4}(1+\frac{dy}{dx})\right)\\
\frac{dy}{dx} &= 2\cdot\left(\frac{\pi}{4}+\frac{\pi}{4}\frac{dy}{dx}\right)\\
\frac{dy}{dx} &= \frac{\pi}{2}+\frac{\pi}{2}\frac{dy}{dx}\\
\frac{dy}{dx}\left(1-\frac{\pi}{2}\right) &= \frac{\pi}{2}\\
\frac{dy}{dx} &= \frac{\pi/2}{1-\pi/2}=\frac{\pi}{2-\pi}.
\end{aligned}$$

Then the equation is

$$y=\frac{\pi}{2-\pi}\cdot x+1$$

4. Suppose a car is travelling along a straight highway. At time $t=0$, this car is located at $s=0$ with initial velocity 9 ft/s. The position function $s=f(t)$ of the car after t seconds is given by

$$f(t)=t^3-6t^2+9t,\ \ t\geq 0.$$

a) Find the velocity and acceleration after t seconds.

b) For what values of $t>0$, does the car stop?

c) For what values of $t>0$, does the car come back to the original position ($s=0$)? What is the velocity at that moment?

d) Draw the position $s(t)$ vs time graph and the velocity $v(t)$ vs time graph.

e) Draw the speed $|v(t)|$ vs time graph.

f) On what intervals is the speed increasing?

Solution: a) $v(t)=f'(t)=3t^2-12t+9$, and $a(t)=f''(t)=6t-12$.

b) Set velocity equal zero.

$$v(t)=3t^2-12t+9=3(t-1)(t-3)=0$$

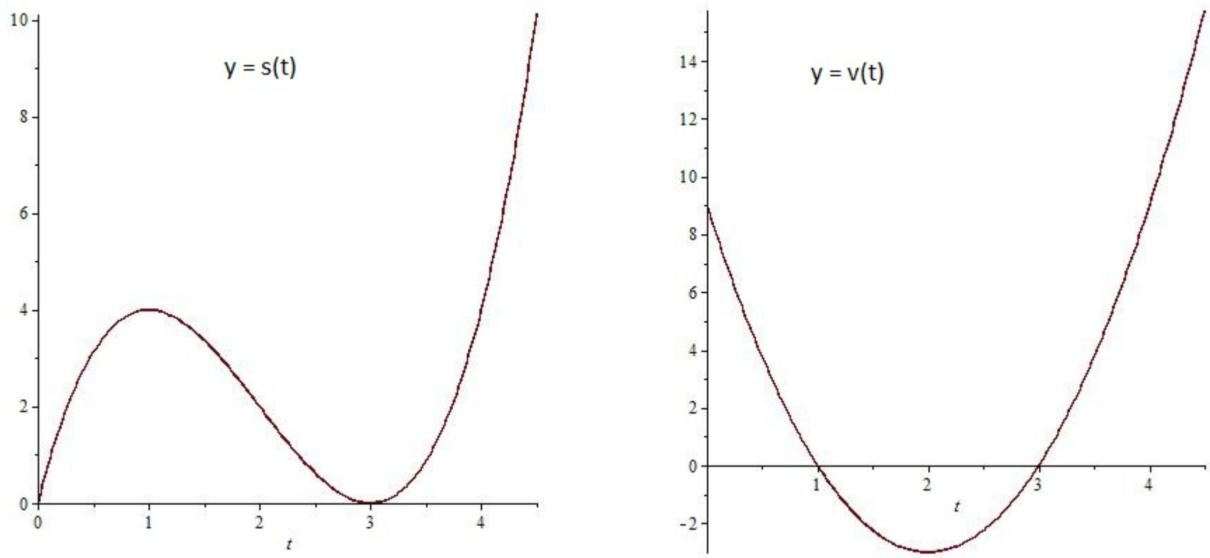
Therefore $t=1,3$.

c) Set position equal zero.

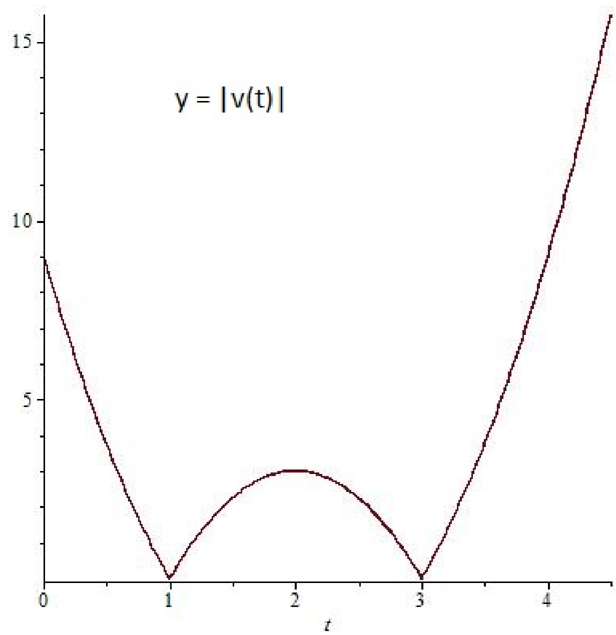
$$s(t) = t^3 - 6t^2 + 9t = t(t-3)^2 = 0$$

Therefore $t = 3$. The velocity is $v(3) = 0$.

d)



e)



f) Use the graph of $y = |v(t)|$. We need to find the point where the slope of tangent is 0.
It is enough to find t such that the slope of tangent of $y = v(t)$ is 0.
The slope of tangent of $y = v(t)$ is $v'(t)$ and setting $v'(t) = 6t - 12 = 0$ gives $t = 2$.
Therefore the intervals where speed is increasing are $(1, 2)$ and $(3, \infty)$.

5. Assume that the variables V, r, and h are all dependent on some variable t. Find $\dfrac{dr}{dt}$ if

$$V = \pi r^2 h$$

SOLUTION: Differentiating with respect to t, we have

$$\begin{aligned}
\frac{dV}{dt} &= \pi\left(2r\frac{dr}{dt}h + r^2\frac{dh}{dt}\right)\\
\frac{dV}{dt} - \pi r^2\frac{dh}{dt} &= \frac{dr}{dt}(2\pi r h)\\
\frac{dr}{dt} &= \frac{\frac{dV}{dt} - \pi r^2\frac{dh}{dt}}{2\pi r h}
\end{aligned}$$

6. Consider the curve $x + y^2 - y = 1$

(a) Find all points on the curve where a vertical tangent line occurs. If it has none, explain why.

(b) Find all points on the curve where a horizontal tangent line occurs. If it has none, explain why.

SOLUTION: $1 + 2y\dfrac{dy}{dx} - \dfrac{dy}{dx} = 0$ so

$$\frac{dy}{dx} = -\frac{1}{2y-1}$$

(a) A vertical tangent line occurs when $\dfrac{dy}{dx}$ is undefined, which occurs when $2y-1=0$, or when $y=\frac{1}{2}$. To find the point on the curve, we plug y back in to find x: $x + (\frac{1}{2})^2 - \frac{1}{2} = 1$, or $x = \frac{5}{4}$. So the point with a vertical tangent line is $(\frac{5}{4}, \frac{1}{2})$.

(b) Since $\dfrac{dy}{dx}$ can never equal 0, there are no horizontal tangent lines.

Below are the problems that were graded and the scoring system that was used for each problem.

1. (b) [5 points] Find the derivative of $e^{x\cos x}$.
0 points – If the student did not use the Chain Rule
2 points – If the student has $e^{x\cos x}$ in his/her solution
2 points – If the student properly found the derivative of $x\cos x$
1 point – Only if the student used the Chain Rule properly and multiplied by the correct derivative of $x\cos x$.

1. (g) [5 points] Find the derivative of $\tan^2 x + \tan(x^2)$.
2 points – If the student found the derivative of $\tan^2 x$ correctly
2 points – If the student found the derivative of $\tan(x^2)$ correctly
1 point – Only if the student properly found the derivative of both terms correctly and added them together in the final answer.

2. [10 points] A differentiable function f satisfies $f(3) = 5$, $f(9) = 7$, $f'(3) = 11$, and $f'(9) = 13$. Find an equation for the tangent line to the curve $y = f(x^2)$ at the point $(x, y) = (3, 7)$.
5 points – If the student found that $\dfrac{dy}{dx} = f'(x^2)\cdot 2x$ [otherwise they get 0 points]
3 points – Only if $\dfrac{dy}{dx}$ was found correctly and the student found the slope of the tangent line to be $m = f'(3^2)\cdot 2(3) = f'(9)\cdot 6 = 13\cdot 6 = 78$
2 points – Only if the slope was found correctly and the equation of the tangent line is correct. If the student writes the equation in point-slope form, s/he receives full credit; s/he does not need to write it in slope-intercept form

6. (b) [10 points] Consider the curve $x + y^2 - y = 1$. Find all points on the curve where a horizontal tangent line occurs. If it has none, explain why.
1 point – If the student found the derivative of x correctly
1 point – If the student found the derivative of y^2 correctly
1 point – If the student found the derivative of y correctly
1 point – If the student found the derivative of 1 correctly

1 point – If the student found all derivatives correctly and has $1 + 2y\frac{dy}{dx} - \frac{dy}{dx} = 0$

2 points – If the student correctly solved for $\frac{dy}{dx}$: $\frac{dy}{dx} = \frac{1}{1-2y}$

3 points – If the student said there are no points where horizontal tangents lines occur with proper reasoning; proper reasoning would be saying that $\frac{dy}{dx}$ can never equal 0