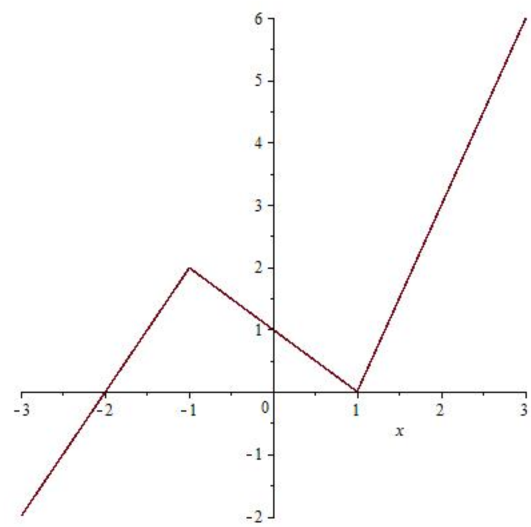
Math 180 Written Homework Solutions

Assignment #4

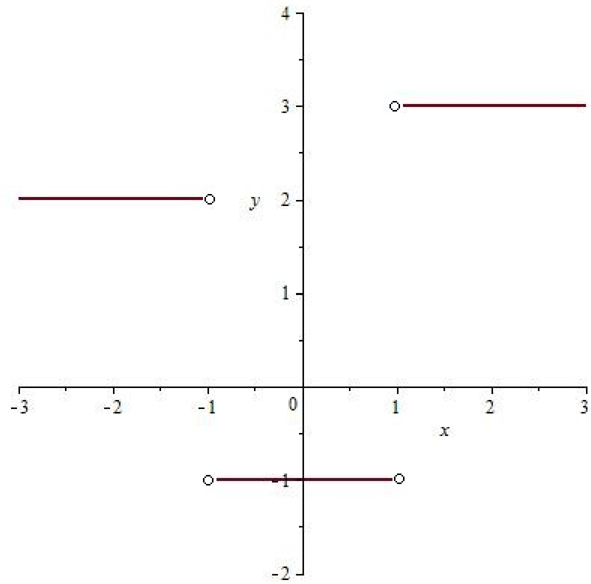
Due **Tuesday, September 23rd** at the beginning of your discussion class.

Directions. You are welcome to work on the following problems with other MATH 180 students, but your solutions must be hand-written, by your own hand, representing your understanding of the material. Word-by-word copying from another student or any other source is unacceptable. Any work without the proper justification will receive no credit. The list of problem solutions is to be submitted to your TA at the beginning of the discussion class listed above. No late homework will be accepted.

1. The following graph represents the graph of some function $f(x)$. Sketch a graph of $f'(x)$ on the interval $-3 \leq x \leq 3$. Note: The vertices of the graph of f and the endpoints of the figure all have integer coordinates.



SOLUTION:



2. Write the equation of the tangent line to $f(x) = 4x^3 + 2x - 1$ at $x = -1$.

   SOLUTION: $f'(x) = 12x^2 + 2$. The slope of the tangent line is $f'(-1) = 12 + 2 = 14$. The point on the tangent line is $(-1, f(-1)) = (-1, -7)$. The tangent line equation is

$$y + 7 = 14(x + 1).$$

3. If the tangent line to $y = f(x)$ at $(4, 3)$ passes through the point $(0, 2)$, then

   (a) find $f(4)$ and $f'(4)$; and

   (b) write the equation of that tangent line.

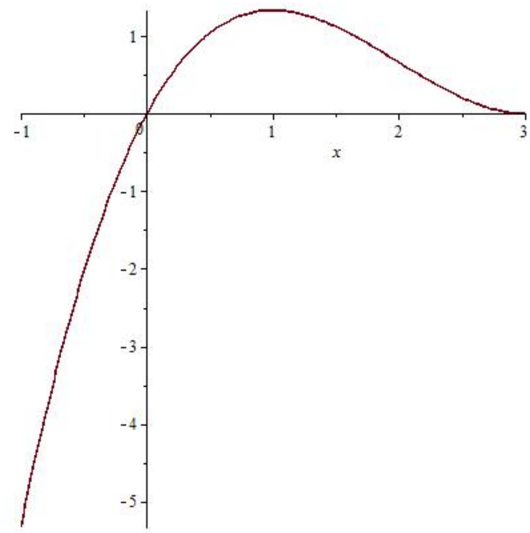
   SOLUTION: (a) $f(4) = 3$ and $f'(4) = \dfrac{3-2}{4-0} = \dfrac{1}{4}$.

   (b)

$$y - 3 = \frac{1}{4}(x - 4)$$

4. Sketch the graph of a function g for which $g(0) = 0$, $g'(0) = 3$, $g'(1) = 0$, and $g'(2) = -1$.

   SOLUTION:



5. Let $f(x) = |\sin x|$. Use the definition of the derivative to find $f'(0)$ or show that $f'(0)$ does not exist.

   SOLUTION:

$$f'(0) = \lim_{h\to 0} \frac{|\sin(0+h)| - |\sin 0|}{h} = \lim_{h\to 0} \frac{|\sin h|}{h}.$$

   If $h > 0$, then $|\sin h| = \sin h$, so

$$\lim_{h\to 0+} \frac{|\sin h|}{h} = \lim_{h\to 0+} \frac{\sin h}{h} = 1.$$

   If $h < 0$, then $|\sin h| = -\sin h$, so

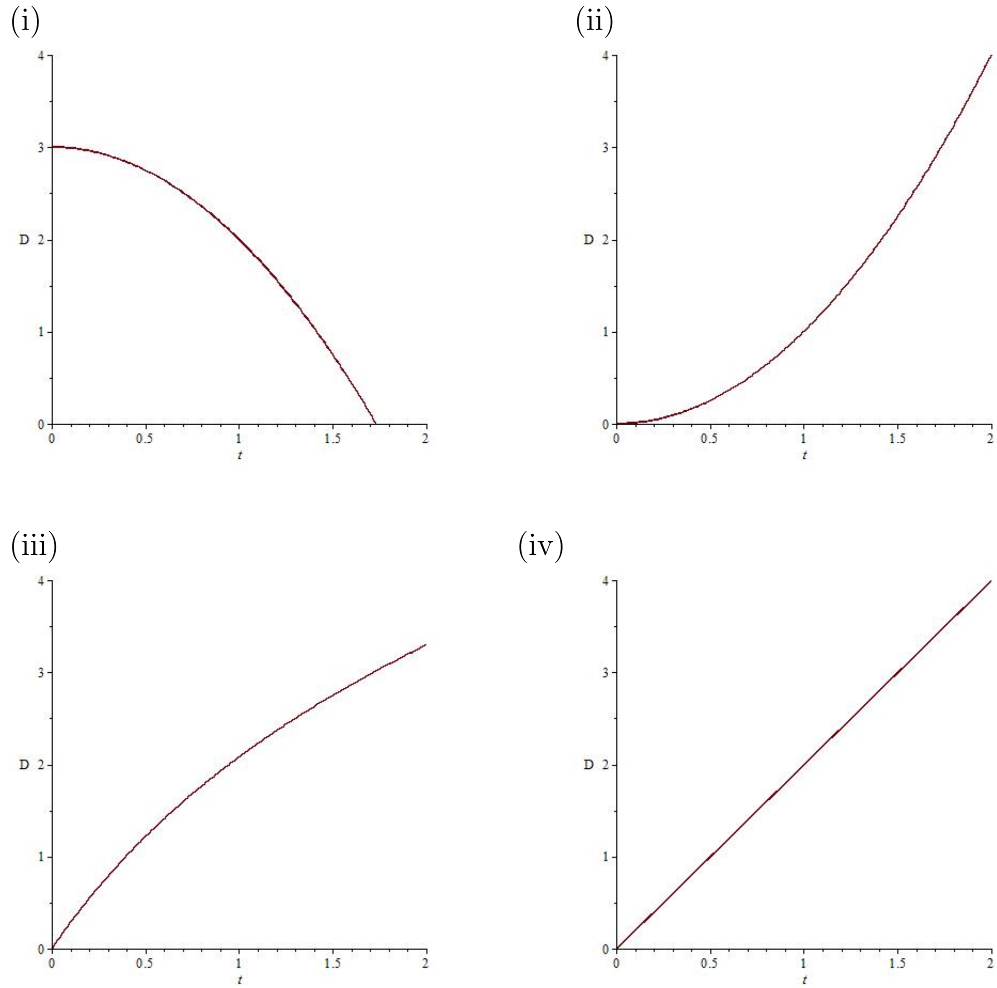
$$\lim_{h\to 0-} \frac{|\sin h|}{h} = \lim_{h\to 0-} \frac{-\sin h}{h} = -1.$$

   Since the left and right hand limits are not equal, $f'(0)$ does not exist.

6. Each of the graphs below represent some function f. Use the graphs to answer the following questions and explain why you chose the answer you did.

   (a) In which graph is f' increasing the fastest?

   (b) In which graph is f' a constant?



Solution: (a) The slopes of the tangent lines in graph (ii) are getting larger the fastest, so f' is increasing fastest in graph (ii).

(b) In graph (iv), the function is linear so the derivative f' is constant.

7. Calculate the following derivatives.

   (a) $x^6 - 4x + 3$

   (b) $\dfrac{\sqrt{x}-1}{\sqrt{x}+1}$

   (c) $x \sin x$

   (d) $\cos^3 x$ [Note: You may not use the "Chain Rule" if you know it.]

   (e) $\dfrac{\tan x}{x}$

   (f) $\dfrac{x \cot x}{\sec x}$

(g) $3x^2 + 5e^x$

SOLUTION:

(a) $6x^5 - 4$

(b) $\dfrac{(\sqrt{x}+1)(\frac{1}{2}x^{-1/2}) - (\sqrt{x}-1)(\frac{1}{2}x^{-1/2})}{(\sqrt{x}+1)^2}$

(c) $\sin x + x\cos x$

(d)

$$\begin{aligned}\frac{d}{dx}\left(\cos^3 x\right) &= \frac{d}{dx}(\cos x\cos x\cos x) = \frac{d}{dx}(\cos x)\cos^2 x + \cos x\frac{d}{dx}(\cos x\cos x) \\ &= -\sin x\cos^2 x + \cos x\left(-\sin x\cos x - \cos x\sin x\right)\end{aligned}$$

(e) $\dfrac{x\sec^2 x - \tan x}{x^2}$

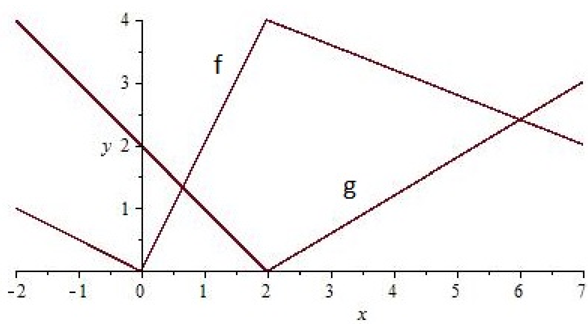
(f) $\dfrac{\sec x(\cot x - x\csc^2 x) - x\cot x(\sec x\tan x)}{\sec^2 x}$

(g) $6x + 5e^x$

8. If f and g are the functions whose graphs are shown, let $u(x) = f(x)\cdot g(x)$ and $v(x) = f(x)/g(x)$. Note: The vertices of the graphs of f and g and the endpoints of the figure all have integer coordinates.

(a) Find $u'(1)$.

(b) Find $v'(5)$.



SOLUTION: For this problem, we have that

$$f(x) = \begin{cases} -\frac{1}{2}x & \text{if } x \le 0 \\ 2x & \text{if } 0 < x \le 2 \\ -\frac{2}{5}(x-2)+4 & \text{if } x > 2 \end{cases} \qquad f'(x) = \begin{cases} -\frac{1}{2} & \text{if } x < 0 \\ 2 & \text{if } 0 < x < 2 \\ -\frac{2}{5} & \text{if } x > 2 \end{cases}$$

$$g(x) = \begin{cases} 2-x & \text{if } x \le 2 \\ \frac{3}{5}(x-2) & \text{if } x > 2 \end{cases} \qquad g'(x) = \begin{cases} -1 & \text{if } x < 2 \\ \frac{3}{5} & \text{if } x > 2 \end{cases}$$

(a)

$$u'(1) = f'(1)\cdot g(1) + f(1)\cdot g'(1) = 2\cdot 1 + 2\cdot(-1) = 0$$

(b)

$$v'(5) = \frac{g(5)\cdot f'(5) - f(5)\cdot g'(5)}{[g(5)]^2} = \frac{\frac{9}{5}\cdot(-\frac{2}{5}) - \frac{14}{5}\cdot\frac{3}{5}}{[\frac{9}{5}]^2} = -\frac{20}{27}$$

9. If $f(x) = -\cos x$, find $f^{(100)}(x)$. Explain how you found your answer.

SOLUTION:

$$\begin{aligned} f'(x) &= \sin x \\ f''(x) &= \cos x \\ f'''(x) &= -\sin x \\ f^{(4)}(x) &= -\cos x = f(x) \\ f^{(5)}(x) &= \sin x = f'(x) \\ &\vdots \\ f^{(100)} &= f^{(4)} = f(x) = -\cos x \end{aligned}$$

10. Find the point(s) on the curve $y = \dfrac{\cos x}{2+\sin x}$ whose x-coordinate satisfies $0 \le x \le 2\pi$ at which the tangent line is horizontal.

SOLUTION:

$$y' = \frac{(2+\sin x)(-\sin x) - (\cos x)(\cos x)}{(2+\sin x)^2} = \frac{-2\sin x - \sin^2 x - \cos^2 x}{(2+\sin x)^2} = \frac{-2\sin x - 1}{(2+\sin x)^2}$$

The tangent lines are horizontal if $y' = 0$, or

$$\begin{aligned} -2\sin x - 1 &= 0 \\ \sin x &= -\frac{1}{2} \\ x &= \frac{7\pi}{6}, \frac{11\pi}{6}. \end{aligned}$$

Therefore the points where the tangent lines are horizontal are

$$\left(\frac{7\pi}{6}, -\frac{\sqrt{3}}{3}\right) \qquad \left(\frac{11\pi}{6}, \frac{\sqrt{3}}{3}\right)$$

Below are the problems that were graded and the scoring system that was used for each problem.

2. [10 points] Write the equation of the tangent line to $f(x) = 4x^3 + 2x - 1$ at $x = -1$.

2 points – If the student found the point on the tangent line $(-1, -7)$

3 points – If the student properly found $f'(x)$

5 points – If the student properly found $f'(x)$ and the slope $f'(-1) = 14$

10 points – Only if the student found $f'(-1)$ correctly and the point can they receive credit for the equation of the line.

7(c). [5 points] Calculate the following derivative. $x \sin x$

0 points – If the student did not use the product rule

1 point – correctly found the derivative of x

1 point – correctly found the derivative of $\sin x$

5 points – Only if a student got both the derivative of x and $\sin x$ correct and applied the product rule correctly.

10. [10 points] Find the point(s) on the curve $y = \dfrac{\cos x}{2 + \sin x}$ whose x-coordinate satisfies

$0 \leq x \leq 2\pi$ at which the tangent line is horizontal.

2 points – If the student calculated the derivative using the quotient rule, but some parts of it are incorrect

5 points – If the student correctly calculated the derivative

8 points – Only if the derivative is correct and they knew to set the numerator equal to 0

10 points – Only if the derivative is correct, they set the numerator equal to 0, and they got the correct points.

9 points – If the students only found the x-values of the points and did not find the y-values