Math 180 Written Homework Solutions

Assignment #3

Due **Tuesday, September 16th** at the beginning of your discussion class.

Directions. You are welcome to work on the following problems with other MATH 180 students, but your solutions must be hand-written, by your own hand, representing your understanding of the material. Word-by-word copying from another student or any other source is unacceptable. Any work without the proper justification will receive no credit. The list of problem solutions is to be submitted to your TA at the beginning of the discussion class listed above. No late homework will be accepted.

In a limit problem where the limit does not exist because of an infinite limit, determine if the limit is $+\infty$ or $-\infty$ and justify your answer.

1. Let $f(x) = \dfrac{\sqrt[3]{x^6+729}}{4x^2+\sqrt{2x^4+64}}$.

   (a) Find the domain of f and state the values where f is continuous. Justify your answers.

   (b) Find $\displaystyle\lim_{x\to-\infty} f(x)$.

   (c) Find $\displaystyle\lim_{x\to+\infty} f(x)$.

   (d) Find all of the horizontal asymptotes of $f(x)$. If it has none, explain why not.

   (e) Find all of the vertical asymptotes of $f(x)$. If it has none, explain why not.

   Solution: (a) Since $4x^2+\sqrt{2x^4+64} > 0$ for any x, the domain is all real numbers. Because the numerator is continuous for all reals, and the denominator is defined and continuous for all reals, the quotient, $f(x)$, is continuous for all real numbers.

   (b) & (c) For any x, we know that $\sqrt[3]{x^6} = x^2$ and $\sqrt{x^4} = x^2$.

$$\begin{aligned}
\frac{\sqrt[3]{x^6+729}}{4x^2+\sqrt{2x^4+64}} &= \frac{\sqrt[3]{x^6+729}}{4x^2+\sqrt{2x^4+64}}\cdot\frac{x^2}{x^2} = \frac{\sqrt[3]{x^6+729}}{4x^2+\sqrt{2x^4+64}}\cdot\frac{\sqrt[3]{x^6}}{x^2} \\
&= \frac{\sqrt[3]{1+\frac{729}{x^6}}}{4+\sqrt{2+\frac{64}{x^4}}}
\end{aligned}$$

   Then

$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty}\frac{\sqrt[3]{1+\cancel{\frac{729}{x^6}}}}{4+\sqrt{2+\cancel{\frac{64}{x^4}}}} = \frac{\sqrt[3]{1}}{4+\sqrt{2}} = \frac{1}{4+\sqrt{2}}$$

$$\lim_{x\to+\infty} f(x) = \lim_{x\to+\infty}\frac{\sqrt[3]{1+\cancel{\frac{729}{x^6}}}}{4+\sqrt{2+\cancel{\frac{64}{x^4}}}} = \frac{\sqrt[3]{1}}{4+\sqrt{2}} = \frac{1}{4+\sqrt{2}}$$

(d) The only horizontal asymptote is $y = \dfrac{1}{4+\sqrt{2}}$.

(e) Since f is defined everywhere, there are no vertical asymptotes.

2. Evaluate the following limits (justify your answers!):

$$\lim_{x\to-\infty} \frac{1}{\arctan x} \qquad\qquad \lim_{x\to+\infty} \frac{1}{\arctan x}.$$

SOLUTION: From the graph of $\arctan(x)$, we know that

$$\lim_{x\to-\infty} \arctan x = -\frac{\pi}{2} \qquad\qquad \lim_{x\to+\infty} \arctan x = \frac{\pi}{2}$$

Therefore

$$\lim_{x\to-\infty} \frac{1}{\arctan x} = -\frac{2}{\pi} \qquad\qquad \lim_{x\to+\infty} \frac{1}{\arctan x} = \frac{2}{\pi}$$

3. Evaluate the following limits (justify your answers!):

$$\lim_{x\to-\infty} e^{-x/10} \qquad\qquad \lim_{x\to+\infty} e^{-x/10}.$$

SOLUTION: The graph of $e^{-x/10}$ is a horizontal stretch of the graph e^{-x}, so both have the same end behavior. We know that $\lim_{x\to-\infty} e^{-x} = +\infty$ and $\lim_{x\to+\infty} e^{-x} = 0$, so

$$\lim_{x\to-\infty} e^{-x/10} = +\infty \qquad\qquad \lim_{x\to+\infty} e^{-x/10} = 0$$

4. Evaluate the following limits (justify your answers!):

$$\lim_{x \to -\infty} \frac{x}{\sqrt{x^2+1}} \qquad\qquad \lim_{x \to +\infty} \frac{x}{\sqrt{x^2+1}}.$$

SOLUTION: If $x > 0$, then $\sqrt{x^2} = x$ so

$$\lim_{x \to +\infty} \frac{x}{\sqrt{x^2+1}} \cdot \frac{\frac{1}{x}}{\frac{1}{\sqrt{x^2}}} = \lim_{x \to +\infty} \frac{1}{\sqrt{1+\cancel{\frac{1}{x^2}}}} = 1$$

If $x < 0$, then $\sqrt{x^2} = -x$ so

$$\lim_{x \to -\infty} \frac{x}{\sqrt{x^2+1}} \cdot \frac{-\frac{1}{x}}{\frac{1}{\sqrt{x^2}}} = \lim_{x \to -\infty} \frac{-1}{\sqrt{1+\cancel{\frac{1}{x^2}}}} = -1$$

5. Find all horizontal asymptotes of $f(x) = \dfrac{\sin^2 x}{x}$.

SOLUTION: We know $-1 \le \sin x \le 1$ so $0 \le \sin^2 x \le 1$. If $x > 0$, then

$$0 \le \frac{\sin^2 x}{x} \le \frac{1}{x}$$

Since $\lim\limits_{x \to +\infty} 0 = \lim\limits_{x \to +\infty} \dfrac{1}{x} = 0$, by the Squeeze Theorem $\lim\limits_{x \to +\infty} \dfrac{\sin^2 x}{x} = 0$. Therefore $y = 0$ is a horizontal asymptote.

If $x < 0$, then

$$0 \ge \frac{\sin^2 x}{x} \ge \frac{1}{x}$$

Since $\lim\limits_{x \to -\infty} 0 = \lim\limits_{x \to -\infty} \dfrac{1}{x} = 0$, by the Squeeze Theorem $\lim\limits_{x \to -\infty} \dfrac{\sin^2 x}{x} = 0$. Therefore $y = 0$ is a horizontal asymptote in the negative direction too.

6. Consider the function $g(x) = \cos\left(e^{\sqrt{x}}\right)$.

(a) State the domain of g.

(b) State the values (if any) in the domain of g where g is discontinuous. Justify your answer.

SOLUTION: (a) The domain is $[0, +\infty)$.

(b) Since e^x is continuous everywhere, then $e^{\sqrt{x}}$ is continuous on the domain of $\sqrt{x}$ which is $[0, +\infty)$. Similarly, since $\cos x$ is continuous everywhere, then $\cos\left(e^{\sqrt{x}}\right)$ is continuous on the domain of $e^{\sqrt{x}}$, which is $[0, +\infty)$.

7. Find the values at which the function

$$f(x) = \begin{cases} 2x+1 & \text{if } x \leq -1 \\ 3x & \text{if } -1 < x < 1 \\ -5x+7 & \text{if } 1 \leq x < 2 \\ 2x-7 & \text{if } x > 2 \end{cases}$$

is discontinuous. Sketch a graph of f to verify your discontinuities.

SOLUTION: The only possibilities for discontinuities for f occur at $x = -1, 1, 2$ since the function is linear for all other values.

At $x = -1$:

$$\begin{aligned} \lim_{x \to -1^-} f(x) &= \lim_{x \to -1^-} (2x+1) = 2(-1)+1 = -1 \\ \lim_{x \to -1^+} f(x) &= \lim_{x \to -1^+} (3x) = 3(-1) = -3 \end{aligned}$$

so f is discontinuous at $x = -1$.

At $x = 1$:

$$\begin{aligned} \lim_{x \to 1^-} f(x) &= \lim_{x \to 1^-} (3x) = 3(1) = 3 \\ \lim_{x \to 1^+} f(x) &= \lim_{x \to 1^+} (-5x+7) = -5(3)+7 = -8 \end{aligned}$$

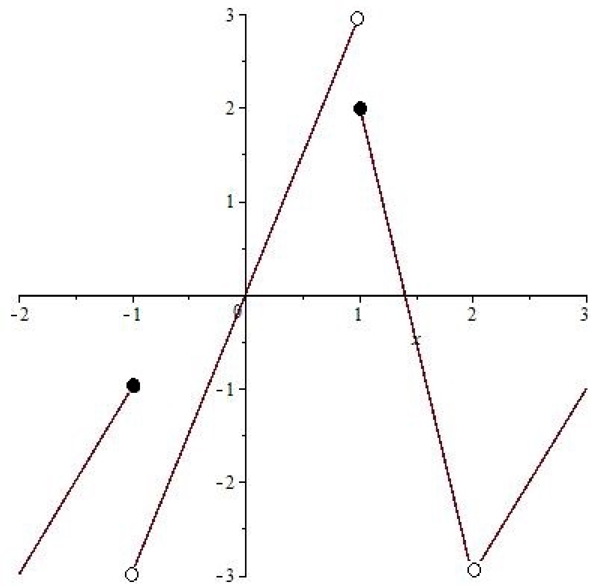
so f is discontinuous at $x = 1$.

At $x = 2$:

$$\begin{aligned} \lim_{x \to 2^-} f(x) &= \lim_{x \to 2^-} (-5x+7) = -5(2)+7 = -3 \\ \lim_{x \to 2^+} f(x) &= \lim_{x \to 1^+} (2x-7) = 2(2)-7 = -3 \end{aligned}$$

however $f(2)$ is undefined at $x = 2$ so f is discontinuous at $x = 2$.

Therefore f is continuous at all values of x except $x = -1, 1, 2$.



8. Find the derivatives of the following functions using the definition of the derivative (i.e. the limit of a difference quotient).

   (a) $x^3 + 2x$

   (b) $\dfrac{1}{\sqrt{t-3}}$

SOLUTION: (a)

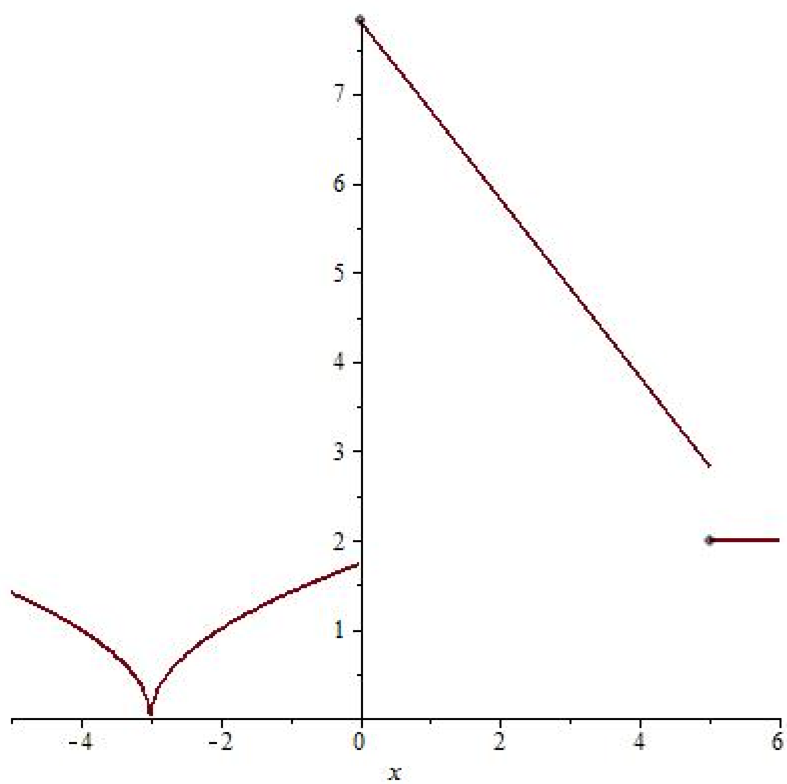
$$\begin{aligned}
\frac{d}{dx}\left(x^3+2x\right) &= \lim_{h\to 0}\frac{[(x+h)^3+2(x+h)]-[x^3+2x]}{h} \\
&= \lim_{h\to 0}\frac{\cancel{x^3}+3x^2h+3xh^2+h^3+\cancel{2x}+2h-\cancel{x^3}-\cancel{2x}}{h} \\
&= \lim_{h\to 0}\frac{\cancel{h}(3x^2+3xh+h^2+2)}{\cancel{h}} = \lim_{h\to 0}(3x^2+3xh+h^2+2) \\
&= 3x^2+2.
\end{aligned}$$

(b)

$$
\begin{aligned}
\frac{d}{dt}\left(\frac{1}{\sqrt{t-3}}\right) &= \lim_{h\to 0}\frac{\dfrac{1}{\sqrt{t+h-3}}-\dfrac{1}{\sqrt{t-3}}}{h} \\
&= \lim_{h\to 0}\frac{\dfrac{\sqrt{t-3}}{\sqrt{t+h-3}\sqrt{t-3}}-\dfrac{\sqrt{t+h-3}}{\sqrt{t+h-3}\sqrt{t-3}}}{h} \\
&= \lim_{h\to 0}\frac{\sqrt{t-3}-\sqrt{t+h-3}}{h\sqrt{t+h-3}\sqrt{t-3}}\cdot\frac{\sqrt{t-3}+\sqrt{t+h-3}}{\sqrt{t-3}+\sqrt{t+h-3}} \\
&= \lim_{h\to 0}\frac{(t-3)-(t+h-3)}{h\sqrt{t+h-3}\sqrt{t-3}(\sqrt{t-3}+\sqrt{t+h-3})} \\
&= \lim_{h\to 0}\frac{-1}{\sqrt{t+h-3}\sqrt{t-3}(\sqrt{t-3}+\sqrt{t+h-3})} \\
&= -\frac{1}{(t-3)\cdot 2\sqrt{t-3}} = -\frac{1}{2}(t-3)^{-3/2}
\end{aligned}
$$

9. Sketch the graph of a function $f(x)$ that is differentiable at every value of x except $x=-3,0,5$. Explain why you drew your graph the way you did.

SOLUTION:



At $x=-3$ there is a cusp so f is not differentiable there. At $x=0$ and $x=5$, f is not continuous so f cannot be differentiable

[Note: This is not the only such function that satisfies the conditions of the problem.]

Below are the problems that were graded and the scoring system that was used for each problem.

1. [15 points] Let $f(x) = \dfrac{\sqrt[3]{x^6 + 729}}{4x^2 + \sqrt{2x^4 + 64}}$.

(a) Find the domain of f and state the values where f is continuous. Justify your answers.

(b) Find $\lim\limits_{x \to -\infty} f(x)$.

(c) Find $\lim\limits_{x \to +\infty} f(x)$.

(d) Find all of the horizontal asymptotes of $f(x)$. If it has none, explain why not.

(e) Find all of the vertical asymptotes of $f(x)$. If it has none, explain why not.

(a) 1 point – If a student has the correct domain and intervals of continuity

3 points – Only if the student has the correct domain and intervals of continuity and justifies both

(b) & (c) [each is worth 4 points]

1 point – If the student knows to divide the numerator and denominator by x^2, but does not correctly divide by it

3 points – If the student properly divides the numerator and denominator by x^2, but does not evaluate the limit correctly

4 points – If the student properly divides the numerator and denominator by x^2, and then evaluates the limit correctly

(d) & (e) [each is worth 2 points]

0 points – If a student's answers to parts (a) and (b)/(c) are wrong

1 point – If a student has the correct value(s) for the asymptotes coming from parts (a) and (b)/(c), but does not write them in equation form (i.e. if a student says the horizontal asymptote is 2 and not $y = 2$)

2 points – If a student has the correct equations of the asymptotes coming from parts (a) and (b)/(c)

4. Evaluate the following limits (justify your answers!):

$$\lim_{x \to -\infty} \frac{x}{\sqrt{x^2+1}} \qquad\qquad \lim_{x \to +\infty} \frac{x}{\sqrt{x^2+1}}.$$

[10 points] $\lim\limits_{x \to -\infty} f(x)$

2 points – If the student knows to divide the numerator and denominator by x, but does not do it correctly

4 points maximum – If the student does not properly introduce the negative sign coming from $-x = \sqrt{x^2}$ for $x < 0$ when dividing numerator and denominator by x

8 points – If the student properly divides the numerator and denominator by x (with the introduction of the negative sign), but does not evaluate the limit correctly

10 points – If the student properly divides the numerator and denominator by x (with the introduction of the negative sign), and then evaluates the limit correctly

DO NOT GRADE THE LIMIT $\lim_{x \to +\infty} f(x)$.

8. Find the derivatives of the following functions using the definition of the derivative (i.e. the limit of a difference quotient).

(a) $x^3 + 2x$

(b) $\dfrac{1}{\sqrt{t-3}}$

[10 points] (a)

0 points – If a student does not use the definition of the derivative

3 points – If the student properly sets up the limit definition of the derivative using $f(x) = x^3 + 2x$, but does not continue correctly from here

8 points – If a student properly sets up the limit definition of the derivative and performs the necessary algebra to put the difference quotient in a form where the limit can be computed

10 points – If a student properly sets up the limit definition, performs the correct algebra, and then computes the limit.

DO NOT GRADE (b).