Math 180 Written Homework

Assignment #10

Due **Tuesday, December 2nd** at the beginning of your discussion class.

Directions. You are welcome to work on the following problems with other MATH 180 students, but your solutions must be hand-written, by your own hand, representing your understanding of the material. Word-by-word copying from another student or any other source is unacceptable. Any work without the proper justification will receive no credit. The list of problem solutions is to be submitted to your TA at the beginning of the discussion class listed above. No late homework will be accepted.

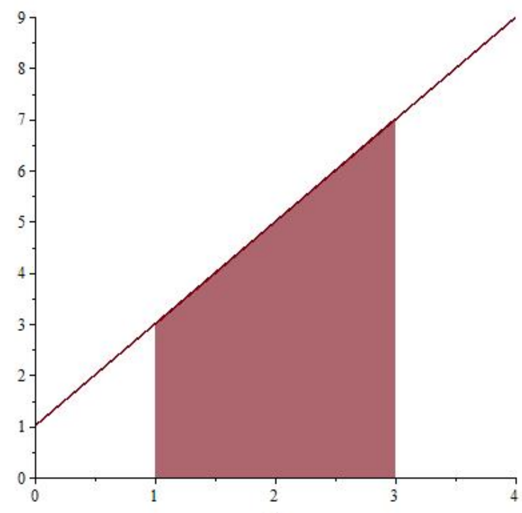
1. For each definite integral draw a picture of the net area the integral is representing. Then calculate the integral (using any method you choose).

   (a) $\displaystyle\int_1^3 (2x+1)\ dx$

   (b) $\displaystyle\int_{-1}^1 (-2|x|+1)\ dx$

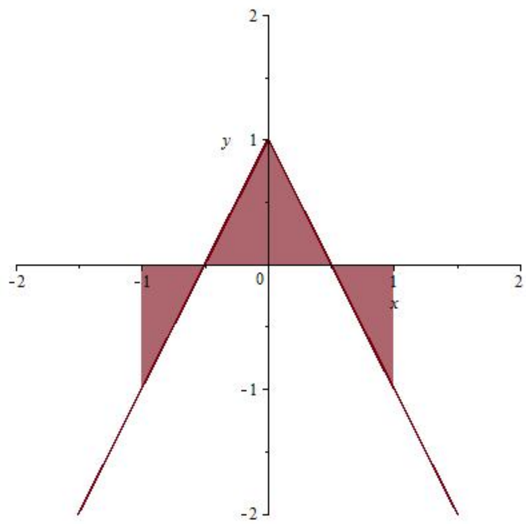
   (c) $\displaystyle\int_0^3 \sqrt{9-t^2}\ dt$

   Solution: (a)



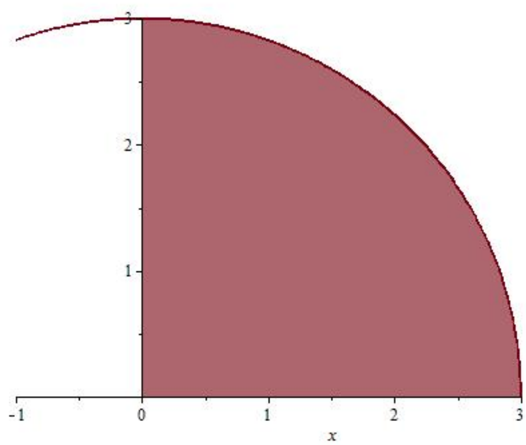
$$\int_1^3 (2x+1)\ dx = \left(x^2+x\right)\Big|_1^3 = 12-2=10$$

(b)



Since the function is even (noting the symmetry with respect to the y-axis), we can see the integral is 0.

(c)



The area sought is a quarter of a circle of radius 3, so

$$\int_0^3 \sqrt{9-t^2}\, dt = \frac{1}{4}\pi(3)^2 = \frac{9}{4}\pi.$$

2. Suppose that $f(x) \geq 0$ on $[0,3]$, $f(x) \leq 0$ on $[3,7]$, $\int_0^3 f(x)\, dx = 4$ and $\int_3^7 f(x)\, dx = -10$.

(a) Sketch a possible graph of f on the interval $[0,7]$.

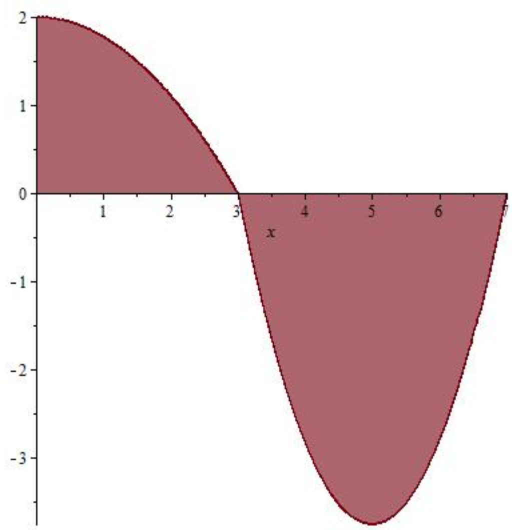
(b) Find $\int_0^7 f(x)\, dx$.

(c) Using your graph from (a), sketch $|f(x)|$ on $[0,7]$.

(d) Find $\int_0^7 |f(x)|\, dx$

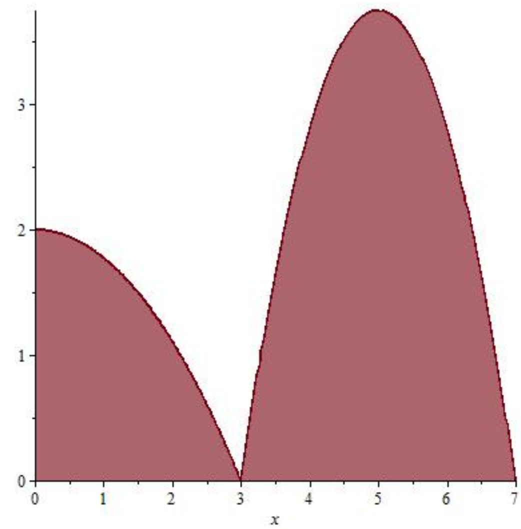
(e) Find $\int_0^7 (3f(x) - 2|f(x)|)\, dx$

SOLUTION: (a) One possible graph is:



(b) $\int_0^7 f(x)\, dx = 4 + (-10) = -6$

(c)



(d) $\int_0^7 |f(x)|\,dx = 4 + 10 = 14$

(e)

$$\int_0^7 (3f(x) - 2|f(x)|)\,dx = 3\int_0^7 f(x)\,dx - 2\int_0^7 |f(x)|\,dx = 3(-6) - 2(14) = -46$$

3. Let $f(x) = \sqrt{24 - 2x - x^2}$.

   (a) What is the domain of $f(x)$?

   (b) Without using a calculator, graph $f(x)$. [Hint: Complete the square.]

   (c) Using geometry, calculate $\int_{-6}^{4} f(x)\,dx$. [Note: You may not use the Fundamental Theorem of Calculus; you must use geometry.]

SOLUTION: (a) The domain is $24 - 2x - x^2 \geq 0$, or $(x+6)(x-4) \geq 0$ which means $-6 \leq x \leq 4$.

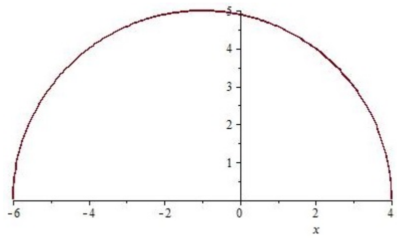
(b) If we complete the square, we have

$$\begin{aligned} 24 - 2x - x^2 &= -(x^2 + 2x - 24) \\ &= -(x^2 + 2x + 1 - 1 - 24) \\ &= 25 - (x+1)^2 \end{aligned}$$

so $f(x) = \sqrt{25-(x+1)^2}$. We can rewrite this as

$$\begin{aligned} y &= \sqrt{25-(x+1)^2} \\ y^2 &= 25-(x+1)^2 \\ (x+1)^2+y^2 &= 25 \end{aligned}$$

This means $f(x)$ is the top half of the circle centered at $(-1,0)$ with radius 5.



(c)

$$\int_{-6}^{4} \sqrt{24-2x-x^2}\, dx = \frac{1}{2}\pi(5)^2 = \frac{25\pi}{2}.$$

4. Let $F(x) = \displaystyle\int_{x^3}^{5} \left(\cos^2 t - te^t\right)\, dt$. Find $F'(x)$.

SOLUTION: Let $f(t) = \cos^2 t - te^t$ and assume $G(t)$ is an antiderivative of $f(t)$; that is $G'(t) = f(t)$. Then

$$\int_{x^3}^{5} \left(\cos^2 t - te^t\right)\, dt = G(x^3) - G(5)$$

Therefore

$$\begin{aligned} \frac{d}{dx}\left(\int_{x^3}^{5} \left(\cos^2 t - te^t\right)\, dt\right) &= \frac{d}{dx}\left(G(x^3) - G(5)\right) \\ &= G'(x^3)\cdot 3x^2 - G'(5)\cdot 0 \\ &= \left(\cos^2(x^3) - x^3e^{x^3}\right)\cdot 3x^2. \end{aligned}$$

5. Calculate the following integrals.

(a) $\displaystyle\int \sin x \cos x \; dx$

(b) $\displaystyle\int \sin^2 t \; dt$ [Hint: $1 - 2\sin^2 t = \cos(2t)$]

(c) $\displaystyle\int_0^{\sqrt{\pi}} xe^{x^2} \; dx$

(d) $\displaystyle\int_{e^2}^{e^4} \frac{3}{x \ln x} \; dx$

(e) $\displaystyle\int x\sqrt{x+2} \; dx$

SOLUTION: (a) If $u = \sin x$, then $du = \cos x \; dx$, so

$$\int \sin x \cos x \; dx = \int u \; du = \frac{1}{2}u^2 + C = \frac{1}{2}\sin^2 x + C.$$

(b) Using the hint

$$\sin^2 t = \frac{1}{2} - \frac{1}{2}\cos(2t).$$

Then

$$\begin{aligned} \int \sin^2 t \; dt &= \int \left(\frac{1}{2} - \frac{1}{2}\cos(2t)\right) \; dt \\ &= \frac{1}{2}t - \frac{1}{2}\int \cos(2t) \; dt. \end{aligned}$$

For this integral, use $u = 2t$ so $du = 2 \; dt$, or $\frac{1}{2} \; du = dt$. Then

$$\begin{aligned} \int \sin^2 t \; dt &= \frac{1}{2}t - \frac{1}{2} \cdot \frac{1}{2}\int \cos u \; du \\ &= \frac{1}{2}t - \frac{1}{4}\sin u + C \\ &= \frac{1}{2}t - \frac{1}{4}\sin(2t) + C. \end{aligned}$$

(c) If $u = x^2$, then $du = 2x \; dx$, or $\frac{1}{2} \; du = x \; dx$. The new limits of integration are $u = 0$ to $u = \pi$. Then

$$\begin{aligned} \int_0^{\sqrt{\pi}} xe^{x^2} \; dx &= \frac{1}{2}\int_0^{\pi} e^u \; du \\ &= \frac{1}{2}e^u\Big|_0^{\pi} \\ &= \frac{1}{2}(e^{\pi} - 1). \end{aligned}$$

(d) If $u = \ln x$, then $du = \frac{1}{x}\, dx$. The new limits of integration are $u = 2$ to $u = 4$. Then

$$\begin{aligned} \int_{e^2}^{e^4} \frac{3}{x \ln x}\, dx &= \int_2^4 \frac{3}{u}\, du \\ &= 3 \ln |u| \Big|_2^4 \\ &= 3\left(\ln(4) - \ln(2)\right) = 3\ln(2). \end{aligned}$$

(e) If $u = x + 2$, then $u - 2 = x$ and $du = dx$. Then

$$\begin{aligned} \int x\sqrt{x+2}\, dx &= \int (u-2)\sqrt{u}\, du = \int (u-2)\cdot u^{1/2}\, du \\ &= \int \left(u^{3/2} - 2u^{1/2}\right)\, du \\ &= \frac{2}{5}u^{5/2} - \frac{4}{3}u^{3/2} + C \\ &= \frac{2}{5}(x+2)^{5/2} - \frac{4}{3}(x+2)^{3/2} + C. \end{aligned}$$

6. Find two values for c such that $\displaystyle\int_0^c (x^2 - 2x)\, dx = 0$.

Solution: Using the FTC, we have

$$\begin{aligned} \int_0^c (x^2 - 2x)\, dx &= 0 \\ \left(\frac{1}{3}x^3 - x^2\right)\Big|_0^c &= 0 \\ \frac{1}{3}c^3 - c^2 &= 0 \\ c^2\left(\frac{1}{3}c - 1\right) &= 0 \\ c &= 0, 3. \end{aligned}$$

7. Let $y = (x-1)^2$.

(a) Find the average value of y on $[0,3]$.

(b) Find all points in $[0,3]$ where f attains the average value you found in (a).

SOLUTION: (a) The average value is

$$\begin{aligned}
\frac{1}{3-0}\int_0^3 (x-1)^2\,dx &= \frac{1}{3}\int_0^3 \left(x^2-2x+1\right)\,dx \\
&= \frac{1}{3}\left(\frac{1}{3}x^3 - x^2 + x\right)\Big|_0^3 \\
&= \frac{1}{3}(9-9+3) = 1
\end{aligned}$$

(b) We're looking for values x such that $f(x) = 1$.

$$\begin{aligned}
(x-1)^2 &= 1 \\
x-1 &= \pm 1 \\
x &= 0, 2.
\end{aligned}$$

Below are the problems that were graded and the scoring system that was used for each problem.

1. (b) [5 points] Draw a picture of the net area the integral is representing and then calculate the integral using any method you choose.

$$\int_{-1}^{1} (-2|x| + 1)\, dx$$

2 points – If the graph is correct

5 points – Only if the graph is correct AND the student uses a legitimate technique for calculating the integral and is correct (i.e. symmetry, rewriting the absolute value and splitting the integral, etc.)

3. [15 points] Let $f(x) = \sqrt{24 - 2x - x^2}$.

(a) What is the domain of $f(x)$?

(b) Without using a calculator, graph $f(x)$. [Hint: Complete the square.]

(c) Using geometry, calculate $\int_{-6}^{4} f(x)\, dx$. [Note: You may not use the Fundamental Theorem of Calculus; you must use geometry.]

(a) [5 points]

2 points – If the student knows to set $24 - 2x - x^2 \geq 0$

5 points – If the student sets $24 - 2x - x^2 \geq 0$ and correctly finds the domain

(b) [5 points]

2 points – If the student makes progress towards completing the square or plots points

5 points – Only if the student shows correct work towards graphing the function (i.e. completing the square, using $f'(x)$, or plotting points, to name a few) AND has the correct graph

(c) [5 points]

0 points – If the graph in part (b) is incorrect

2 points – If the student has the graph in part (b) correct and has an error when calculating the integral

5 points – If the student has the graph in part (b) correct AND correctly calculates the integral using the area of a circle formula

5. (c) [10 points] Calculate $\int_{0}^{\sqrt{\pi}} xe^{x^2}\, dx$

2 points – If the student tries a u-substitution and computes du correctly for their choice of u, but it is the wrong u-substitution and the rest of the problem is wrong

4 points – If the student chooses $u = x^2$ and correctly finds $du = 2x\ dx$

6 points – If the student chooses $u = x^2$, correctly finds $du = 2x\ dx$, and makes the integral $\int e^u\ du$

8 points – If the student chooses $u = x^2$ and correctly finds $du = 2x\ dx$ and has either $\left.\frac{1}{2}e^u\right|_0^{\pi}$ or $\left.\frac{1}{2}e^{x^2}\right|_0^{\sqrt{\pi}}$. [Note: some students may simply recognize the derivative of e^{x^2} is $2xe^{x^2}$ and can calculate the antiderivative of xe^{x^2} in their heads and should receive full credit to any point in the solution without showing work; however if any part of the answer is wrong and no work is shown, then no credit shall be given.]

10 points – If the student correctly calculates the integral

Note: Deduct 1 point if a student has $du = 2x$ and forgets the dx anywhere in the solution. Only deduct one point if the same error is repeated multiple times, but circle each instance of the error.