Math 180 **Name (Print):** ______________________
Fall 2014
Second Midterm
10/22/2014
Time Limit: 2 Hours

This exam contains 10 pages (including this cover page) and 9 problems. Check to see if any pages are missing. Enter all requested information on the top of this page.

The following rules apply:

- You are expected to abide by the University's rules concerning Academic Honesty.
- You may *not* use your books, notes, or any electronic device including cell phones.
- You must show all of your work. An answer, right or wrong, without the proper justification will receive little to no credit.
- You must complete your work in the space provided. No extra paper will be provided.

Problem	Points	Score
1	20	
2	10	
3	10	
4	10	
5	10	
6	10	
7	10	
8	10	
9	10	
Total:	100	

Circle your instructor.

- Adrovic @ 11am
- Adrovic @ 1pm
- Cabrera
- Dumas
- Kashcheyeva
- Kobotis
- Lowman
- Shin @ 10am
- Shin @ 2pm
- Shulman @ 8am
- Shulman @ 9am

TA Name:______________________

1. (20 points) Find the following derivatives. Do not simplify your answers.

   (a) (5 points) $\dfrac{d}{dx}\left(\tan^{-1}(5^x)\right)$

   (b) (5 points) $\dfrac{d}{dx}\left(\sqrt{e^x+\ln x}\right)$

   (c) (5 points) $\dfrac{d}{dx}\left(\log_{10}(\cos x)\right)$

   (d) (5 points) $\dfrac{d}{dx}\left(e^{\sin(x^2)}\right)$

Solution: (a)

$$\frac{1}{1+(5^x)^2}\cdot 5^x\ln(5)$$

(b)

$$\frac{1}{2}(e^x+\ln x)^{-1/2}\cdot\left(e^x+\frac{1}{x}\right)$$

(c)

$$\frac{1}{\ln(10)}\cdot\frac{1}{\cos x}\cdot(-\sin x)$$

(d)

$$e^{\sin(x^2)}\cdot\cos(x^2)\cdot 2x$$

Grading Rubric: (a)

2 points – If the student correctly finds the derivative of $\tan^{-1}(x)$ and substitutes in 5^x.

2 points – If the student correctly finds the derivative of 5^x.

1 point – If the student correctly uses the Chain Rule and has the correct final answer

(b)

2 points – If the student correctly finds the derivative of $\sqrt{x}$ and substitutes in $e^x+\ln x$

1 point – If the student correctly finds the derivative of $e^x+\ln x$

2 points – If the student correctly uses the Chain Rule and has the correct final answer

(c)

2 points – If the student correctly finds the derivative of $\log_{10} x$ and substitutes in $\cos x$

1 point – If the student correctly finds the derivative of $\cos x$ (the sign must be correct)

2 points – If the student correctly uses the Chain Rule and has the correct final answer

(d)

1 point – If the student correctly finds the derivative of e^x and substitutes in $\sin(x^2)$

1 point – If the student correctly finds the derivative of $\sin x$ and substitutes in x^2

1 point – If the student uses the Chain Rule correctly

1 point – If the student uses the Chain Rule a second time correctly

1 point – If the final answer is correct

2. (10 points) If $x^2 - y + y^3 = 16$, find $\dfrac{dy}{dx}$.

SOLUTION:

$$\begin{aligned} 2x - \frac{dy}{dx} + 3y^2\frac{dy}{dx} &= 0 \\ \frac{dy}{dx}\left(-1+3y^2\right) &= -2x \\ \frac{dy}{dx} &= \frac{-2x}{-1+3y^2} \end{aligned}$$

GRADING RUBRIC:

1 point – If the student correctly finds the derivative of x^2

1 point – If the student correctly finds the derivative of y

1 point – If the student correctly finds the derivative of y^3

1 point – If the student correctly finds the derivative of 16

1 point – If the student has $2x - \dfrac{dy}{dx} + 3y^2\dfrac{dy}{dx} = 0$

When solving for $\dfrac{dy}{dx}$:

5 points – If the student correctly solves the equation for $\dfrac{dy}{dx}$

OR

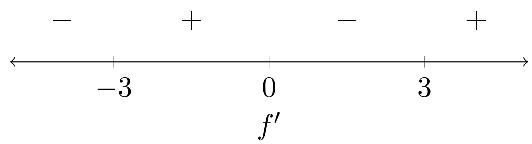
3 points – If the student solves for $\dfrac{dy}{dx}$ and makes a small sign error

OR

0 points – If the student solves for $\dfrac{dy}{dx}$ and makes multiple sign errors or major errors

3. (10 points) Let $f(x) = x^4 - 18x^2$. Find the intervals on which f is increasing and the intervals on which f is decreasing.

SOLUTION: $f'(x) = 4x^3 - 36x = 4x(x^2 - 9) = 0$, so the critical points are $x = 0, \pm 3$.



f is increasing on $(-3, 0)$ and $(3, \infty)$. f is decreasing on $(-\infty, -3)$ and $(0, 3)$.

GRADING RUBRIC:

2 points – If the student computes f' correctly

OR

3 points – If f' is computed correctly and only one of the critical points is found correctly

OR

4 points – If f' is computed correctly and only two of the critical points are found correctly

OR

5 points – If f' is computed correctly and all of the critical points are found correctly

OR

8 points – If f' is computed correctly, all critical points are found, and the sign diagram is completed correctly

OR

10 points – If f' is computed correctly, all critical points are found, the sign diagram is correct, and the intervals are stated correctly

Note: If the student made a minor error when finding the derivative, but continues the rest of the problem correctly, then 1 point was deducted.

4. (10 points) Find the absolute maximum and absolute minimum of $f(x) = \sin x + \cos x$ on $\left[0, \frac{\pi}{2}\right]$.

SOLUTION: $f'(x) = \cos x - \sin x = 0$. The only value where $\sin x = \cos x$ on $\left[0, \frac{\pi}{2}\right]$ is $x = \frac{\pi}{4}$. Now we plug the endpoints and the critical points back into f to find the absolute extrema.

$$\begin{aligned} f(0) &= 1 \\ f\left(\frac{\pi}{2}\right) &= 1 \\ f\left(\frac{\pi}{4}\right) &= \sqrt{2} \end{aligned}$$

The absolute maximum is $\sqrt{2}$ and the absolute minimum is 1.

GRADING RUBRIC:

3 points – If the student correctly finds f'

3 points – If the student correctly solves $f'(x) = 0$

1 point (for each endpoint and the critical point) – If the student evaluates the point for $f(x)$

1 point – If the student comes to the correct conclusion

Note: If the student has a minor error when calculating $f'(x)$, say $f'(x) = \cos x + \sin x$, and proceeds then grade as if correct but deduct 2 points at the end

Note: If the student finds that one or more values other than $x = \frac{\pi}{4}$ is a solution to $f'(x) = 0$, then deduct 2 points.

0 points – If the student uses no calculus to complete the problem

5. (10 points) Let $f(x) = x^7 + x^3 - 2$. Compute $(f^{-1})'(0)$.

SOLUTION: The point $(1, 0)$ is on f. $f'(x) = 7x^6 + 3x^2$, so $f'(1) = 10$. Therefore

$$(f^{-1})'(0) = \frac{1}{f'(1)} = \frac{1}{10}.$$

GRADING RUBRIC:

0 points – If the student only writes the formula

0 points – If the student uses an incorrect method (i.e. implicit differentiation, logarithmic differentiation, etc.)

3 points – If the student solves for $f(x) = 0$ to find that $(1, 0)$ is on f

2 points – If the student states and applies the (incorrect) formula $(f^{-1})'(0) = \dfrac{1}{f(0)}$

OR

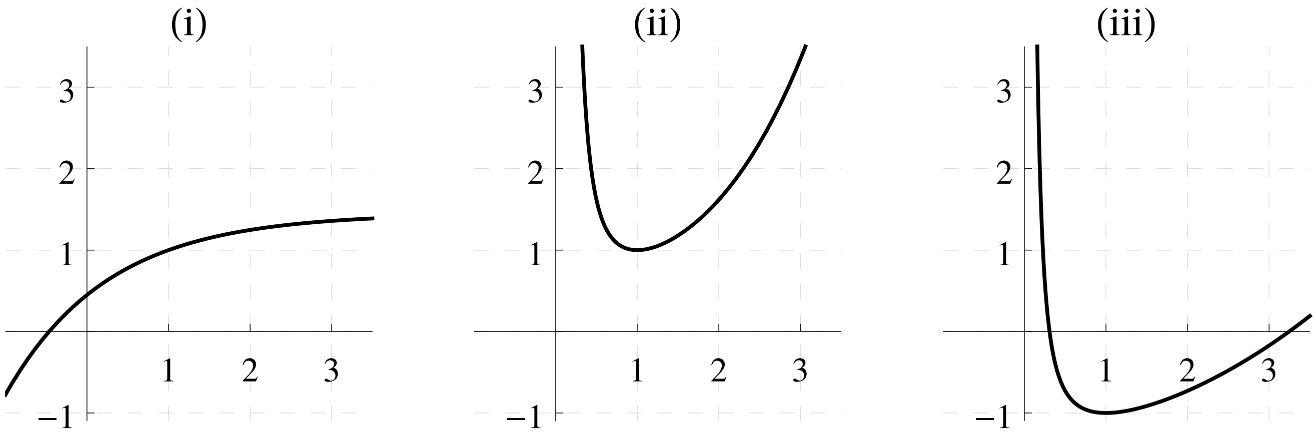
8 points – If the student finds $(1, 0)$ on f and applies the correct formula to find $(f^{-1})'(0)$ but has a small arithmetic error

OR

10 points – If the student finds $(1, 0)$ on f and applies the correct formula with the right answer.

6. (10 points) Consider the function $f(x) = x^{\ln x}$.

   (a) (5 points) Three plots are shown below. One of them is the graph $y = x^{\ln x}$. Which one is it, and why?



   (b) (5 points) Calculate the derivative $f'(x)$. [Your final answer should only involve x; it should not involve y.]

Solution: (a) Since the domain of $y = x^{\ln x}$ is $x > 0$, it cannot be graph (i). Since $y(1) = 1^{\ln(1)} = 1^0 = 1$, it cannot be graph (iii). Therefore $y = x^{\ln x}$ is graph (ii).

(b)

$$\begin{aligned}
y &= x^{\ln x} \\
\ln y &= \ln x \cdot \ln x = (\ln x)^2 \\
\frac{1}{y} \cdot \frac{dy}{dx} &= 2 \ln x \cdot \frac{1}{x} \\
\frac{dy}{dx} &= y \cdot \frac{2 \ln x}{x} \\
\frac{dy}{dx} &= x^{\ln x} \cdot \frac{2 \ln x}{x}
\end{aligned}$$

Grading Rubric: (a)

2 points – If the student correctly explains why the graph of $y = x^{\ln x}$ cannot be (i) or (iii), but not both

5 points – If the student correctly explains why both graphs (i) and (iii) cannot be $y = x^{\ln x}$

(b)

2 points – If the student rewrites $y = x^{\ln x}$ as $\ln y = \ln x \cdot \ln x$

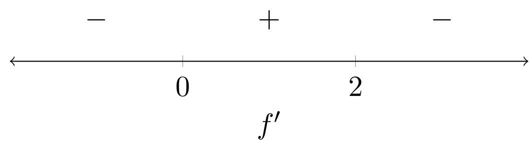
OR

2 points – If the student rewrites $y = x^{\ln x}$ as $y = e^{\ln x \cdot \ln x}$. [Note: If the student just writes $y = e^{\ln(x^{\ln x})}$, then 0 points are awarded; s/he must demonstrate that s/he knows the properties of logarithms]

5 points – Only if the student properly rewrote the equation and found $\frac{dy}{dx}$

7. (10 points) Let $g(x) = x^2 e^{-x}$. Find and classify the critical points of f. (For each critical point, determine if a local maximum, local minimum, or neither exists there.)

SOLUTION: $f'(x) = 2xe^{-x} - x^2e^{-x} = e^{-x}(2x - x^2)$. Since e^{-x} is never equal to 0, the only critical points come from $2x - x^2 = x(2 - x) = 0$, so the critical points are $x = 0$ and $x = 2$. The first derivative sign diagram is



Therefore there is a local minimum at $x = 0$ and a local maximum at $x = 2$.

GRADING RUBRIC:

If the student finds $f'(x)$ correctly then:

3 points – If the student finds $f'(x)$ correctly

1 point – If the student gets to $0 = 2x - x^2$

2 points – If the student finds the critical points $x = 0$ and $x = 2$

1 point (for each critical point) – If some test is applied to classify the critical point

1 point (for each critical point) – If some test is applied to classify the critical point and is done correctly

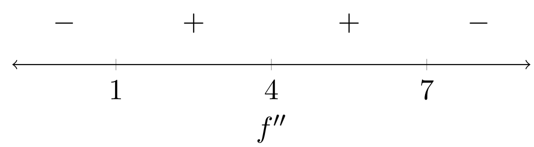
If the student is nearly correct in finding $f'(x)$ (say, $f'(x) = 2xe^{-x} + x^2e^{-x}$) then deduct 4 points and proceed as in the $f'(x)$ correct case above.

If the student is not near finding $f'(x)$ (say $f'(x) = -2xe^{-x}$ or $f'(x) = x^2e^{-x}$) then 0 points are awarded.

8. (10 points) Assume that $f(x)$ is a polynomial and that the second derivative of f is

$f''(x) = -(x-1)(x-4)^2(x-7)$. Find the intervals on which f is concave up and the intervals on which f is concave down. Also, identify all inflection points of f.

Solution: The possible inflection points are $x = 1, 4, 7$. The second derivative sign diagram is



Therefore f is concave up on $(1,4)$ and $(4,7)$ and f is concave down on $(-\infty, 1)$ and $(7, \infty)$. Also, $x = 1$ and $x = 7$ are the inflection points.

Grading Rubric:

For concave up and concave down:

3 points – If 1 or 2 of the intervals of concavity are correct

4 points – If 3 of the intervals of concavity are correct

6 points – If all 4 intervals of concavity are correct

Note: 1 point was deducted if the student wrote $(1,7)$ for concave up and this is the only mistake.

For inflection points, the following points are only awarded if the conclusions match their second derivative sign diagram (which may or may not be correct)

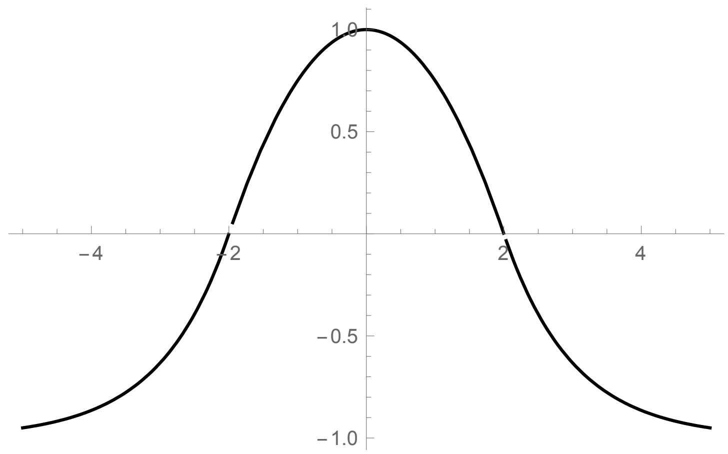
2 points – If 1 or 2 of the inflection points are correct

4 points – If all 3 points are properly labelled as inflection points (which includes NOT labelling $x = 4$ as an inflection point)

9. (10 points) Draw the graph of one function, $f(x)$, that is differentiable on $(-\infty, \infty)$ and satisfies the following properties:

- $f(2) = f(-2) = 0$
- $f'(x) > 0$ on $(-\infty, 0)$ and $f'(x) < 0$ on $(0, \infty)$
- $f''(x) > 0$ on $(-\infty, -2)$, $f''(x) < 0$ on $(-2, 2)$, and $f''(x) > 0$ on $(2, \infty)$

SOLUTION: Here is one solution.



GRADING RUBRIC:

For the first bullet:

2 points – If the graph is drawn through $(-2, 0)$ and $(2, 0)$

For the second bullet (monotonicity):

2 points – If the graph is drawn as increasing on $(-\infty, 0)$ or decreasing on $(0, \infty)$, but not both

4 points – If the graph is drawn as increasing on most of the left side of the y-axis, and drawn as decreasing on most of the right side of the y-axis. [Note: If the student's graph seems to loop up at the ends, then that graph falls into this category.]

5 points – If the graph is drawn as increasing on $(-\infty, 0)$ and decreasing on $(0, \infty)$

For the third bullet (concavity):

1 point – If the graph is drawn as concave up on $(-\infty, 2)$

1 point – If the graph is drawn as concave down on $(2, -2)$

1 point – If the graph is drawn as concave up on $(2, \infty)$

Amendments. If the student makes the following errors, a maximum number of points can be awarded for the full problem:

Maximum 2 points – If a student does not draw a function

Maximum 6 points – If the student's graph has any breaks, corners, or cusps (i.e. if the graph is not differentiable on $(-\infty, \infty)$)

Maximum 6 points – If the student's graph does not have any inflection points