Math 570 / David Dumas / Spring 2013

Problems

This list was last updated on 2013-04-02.

(1) Within $\{cz^{-2}dz^2\} \simeq \mathbb{C}$, determine the subset consisting of Schwarzian derivatives of univalent maps on the upper half-plane.

(2) (a) Generalizing the Schwarzian case, describe all three of the cocycles D, N, S as Darboux derivatives of osculation maps.
   (b) In each case there is a part of the Darboux derivative that does not depend on the function; identify its geometric significance.

(3) Find a necessary and sufficient condition for a holomorphic map $U \to \mathrm{PSL}_2\,\mathbb{C}$ to be realizable as the osculation map of a function. (The condition should be something you can check given the differential of the map.)

(4) Describe as explicitly as possible a map $f : \mathbb{C} \to \hat{\mathbb{C}}$ with $S(f) = z dz^2$. Try to understand its mapping properties geometrically.

(5) Determine the function that relates the modulus of a quadrilateral to the cross ratio of the images of its vertices under the conformal map to the $\mathbb{H}$.

(6) The univalent maps $z + cz^{-1}$ on Δ^{-1} and z^α on $\mathbb{H}$ have simple quasiconformal extensions to $\hat{\mathbb{C}}$ that we discussed in lecture. Does the Ahlfors-Weill extension apply to these univalent maps, and if so, does it recover the same quasiconformal extension?

(7) Show that if a smooth simple closed curve $\gamma \subset \mathbb{C}$ has curvature in the interval $[1 - \epsilon, 1 + \epsilon]$, then it is the image of S^1 under a $K(\epsilon)$-quasiconformal map, where $K(\epsilon) \to 1$ as $\epsilon \to 0$.

(8) Fill in the details necessary to define the modulus of an annulus by conformal mapping, without using the uniformization theorem. Start by proving the existence of a suitable harmonic function with boundary values 0 and 1 on the two boundary components (by the Perron method or otherwise).

(9) Given four points $z_1, z_2, z_3, z_4 \in \hat{\mathbb{C}}$, consider the conformal metric $\rho = \dfrac{|dz|}{|(z - z1)(z - z2)(z - z3)(z - z4)|^{1/2}}$.
   (a) Show that ρ is flat (locally isometric to the Euclidean plane) except at the points z_i which are conical singularities with cone angle π (meaning a neighborhood of each such point is isometric

to a neighborhood of 0 in the quotient $\mathbb{C}/\{z \mapsto -z\}$, where $\mathbb{C}$ has the Euclidean metric).

(b) Identify a configuration of points z_i so that the resulting conformal metric is isometric to the surface of a regular tetrahedron in $\mathbb{R}^3$.

(c) Cutting $\hat{\mathbb{C}}$ along ρ-geodesics joining z_1 to z_2 and z_3 to z_4 gives an annulus with a Euclidean metric. This is the extremal annulus for the problem of separating the sets $\{z_1, z_2\}$ and $\{z_3, z_4\}$. Show that the Teichmüller extremal problem is a special case of this.

(10) Let S^2 denote the unit sphere in $\mathbb{R}^3$. We identify S^2 with $\hat{\mathbb{C}}$ through stereographic projection in order to consider it as a Riemann surface.

We say that a homeomorphism $f : S^2 \to S^2$ *flattens a regular tetrahedron* if there are four points $p_1, \ldots, p_4 \in S^2$ that are vertices of an inscribed regular tetrahedron such that the points $f(p_i)$ are coplanar (i.e. they lie on a great circle on the sphere).

(a) Show that a conformal map cannot flatten a regular tetrahedron.

(b) Show that a 1.0001-quasiconformal map cannot flatten a regular tetrahedron.

(c) Show that for large enough K, there is a K-quasiconformal map that flattens a regular tetrahedron.

(d) Estimate the minimum K such that there exists a K-quasiconformal map flattening a regular tetrahedron.

(11) Every compact Riemann surface of genus 2 is *hyperelliptic*, meaning that it can be described as a 2-fold branched cover of $\hat{\mathbb{C}}$. There are $6 = (2\chi(\hat{\mathbb{C}}) - \chi(S))$ branch points. Which configuration of branch points corresponds to each genus 2 surface described below?

(a) Identify opposite sides of a regular hyperbolic octagon with interior angles $\pi/4$.

(b) Identify opposite sides of a regular hyperbolic decagon with interior angles $2\pi/5$.

(c) Take two regular Euclidean pentagons that share one edge and glue each pair of parallel sides by a translation. (This describes a genus 2 Riemann surface, and a quadratic differential, but the hyperbolic structure is not specified.)

(12) Consider the equations

$$\begin{aligned} z_1 + z_2 + z_3 + z_4 + z_5 &= 0 \\ z_1^2 + z_2^2 + z_3^2 + z_4^2 + z_5^2 &= 0 \\ z_1^3 + z_2^3 + z_3^3 + z_4^3 + z_5^3 &= 0 \end{aligned}$$

Show that:

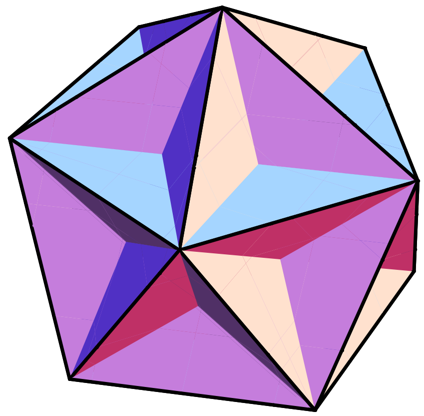

FIGURE 1. The great dodecahedron.

(a) In homogeneous coordinates for $\mathbb{P}^4$, these equations define an irreducible smooth curve of genus 4.
(b) This curve is *trigonal* (a branched cover of $\hat{\mathbb{C}}$ with 3 sheets).
(c) The associated hyperbolic surface is tiled by 24 regular right-angled hyperbolic pentagons.
(d) The associated hyperbolic surface is also tiled by 12 regular hyperbolic pentagons meeting 5 to a vertex.
(e) Each vertex of the (regular, Euclidean) icosahedron has five neighbors, and these form a pentagon. The union of these 12 pentagons is a self-intersecting surface in $\mathbb{R}^3$, the *great dodecahedron*. Ignoring the self-intersections, show that this surface has genus 4. Show that the projection of this surface to a small sphere centered at the origin is a topological model for the algebraic curve described above as a 3-fold covering of $\hat{\mathbb{C}}$.

(13) Let X be a compact Riemann surface of genus $g > 1$. For any measurable set $E \subset X$ there is a subspace $\mathrm{Belt}_1(E) \subset \mathrm{Belt}_1(X)$ consisting of Beltrami differentials supported on E. This subspace projects to a set $D(E) \subset \mathcal{T}(X)$. Clearly if $E = X$ then $D(E) = \mathcal{T}(X)$. Under what circumstances is $D(E)$ contained in a compact subset of $\mathcal{T}(X)$? Consider for example when E is:
- A small disk
- A homotopically nontrivial annulus
- The complement of a homotopically nontrivial annulus
- The complement of a small disk

(14) Consider a *swiss cross* in the plane, i.e. the union of a unit square $[0,1] \times [0,1]$ and its reflections across its four sides. The result is a non-convex dodecagon. Now identify any pair of sides that can

be mapped to one another by a purely vertical or purely horizontal translation.

(a) The result of these identifications is a Riemann surface X. What is its genus?

(b) The quadratic differential dz^2 on the plane descends to a quadratic differential $\phi \in Q(X)$. Determine the number and multiplicities of the zeros.

(c) The (horizontal) foliation of ϕ has singular trajectories that divide X into two flat cylinders. What other directions induce complete cylinder decompositions like this? (Try some small rational multiples of π.) In cases where you find a cylinder decomposition using the swiss cross picture, can you visualize it on the surface X?