Math 570 / David Dumas / Spring 2013

Irreducible Holonomy

Theorem. *The holonomy representation of a $\mathbb{CP}^1$ structure on a compact surface of genus $g > 1$ is irreducible.*

Proof summary. If the holonomy is reducible, the residue theorem shows that the $\mathbb{CP}^1$ structure is actually an affine structure, hence the tangent bundle has degree zero and $g = 1$.

Proof. Suppose on the contrary that X is a Riemann surface of genus $g > 1$ which has a projective structure (f, ρ) such that $\rho : \pi_1 S \to \mathrm{PSL}_2\mathbb{C}$ is reducible.

Without loss of generality we assume that $\rho(\pi_1 S)$ is contained in the stabilizer of $\infty \in \mathbb{CP}^1$, since every reducible representation is conjugate to one of this form. Thus each holonomy element is a complex affine map, of the form $z \mapsto (az + b)$.

By uniformization we have $X \simeq \mathbb{H}/\Gamma$ for a Fuchsian group Γ, so we can regard the developing map f as a meromorphic function on $\mathbb{H}$.

The nonlinearity $N(f) = f''/f'$ is a meromorphic 1-form on $\mathbb{H}$. This form is Γ-invariant since the nonlinearity is unchanged by post-composition with complex affine maps. Thus $N(f)$ descends to a meromorphic 1-form ω on X.

Since f is an immersion into $\mathbb{CP}^1$, the only poles of $N(f)$ are in $f^{-1}(\infty)$. A local calculation shows that each such preimage of ∞ gives a pole with residue -2:

$$f(z) = az^{-1} + b + cz + \cdots$$
$$N(f)(z) = -2z^{-1} - \frac{2c}{a} z + \cdots$$

Therefore, all poles of ω have the same nonzero residue. However, the residues of a meromorphic 1-form on a compact Riemann surface sum to zero. Therefore ω has no poles and $f^{-1}(\infty)$ is empty, i.e. the developing map has image in $\mathbb{C}$.

It follows that (f, ρ) actually induce on X a *complex affine structure*, defined by charts into $\mathbb{C}$ with transition functions in $\{z \mapsto az + b\}$. The desired contradiction is therefore furnished by the following lemma. $\square$

Lemma. *If a compact Riemann surface X admits a complex affine structure, then X has genus 1.*

Proof. The transition functions for the affine structure have constant derivative. Thus the tangent bundle of X can be described by transition functions that are constant. However, any line bundle with constant transition functions has vanishing first Chern class*. Since $c_1(TX) = 2 - 2g$, it follows that $g = 1$. $\square$

Here are some more details on the point (*), that constant transition functions imply vanishing of the Chern class: There is an exact sequence of sheaves on X,

$$0 \to \mathbb{C}^* \to \mathcal{O}^* \to \mathcal{O}^{1,0} \to 0$$

where $\mathbb{C}^*$ is the constant sheaf, $\mathcal{O}^*$ is the sheaf of nonvanishing holomorphic functions, and $\mathcal{O}^{1,0}$ of holomorphic 1-forms. Here the map $\mathbb{C}^* \to \mathcal{O}^*$ is just the inclusion, while $\mathcal{O}^* \to \mathcal{O}^{1,0}$ is the logarithmic derivative, $f \mapsto d\log(f)$.

Writing out the associated long exact sequence in cohomology we find

$$H^1(\mathbb{C}^*) \to H^1(\mathcal{O}^*) \to \mathbb{Z} \to 0 \tag{1}$$

where the constant sheaf $\mathbb{Z}$ is actually the kernel of

$$\left[\mathbb{C} \simeq H^0(\mathcal{O}) \simeq\right] H^1(\mathcal{O}^{1,0}) \longrightarrow H^2(\mathbb{C}^*) \left[\simeq \mathbb{C}^*\right],$$

a map that can be identified with the exponential. (In the bracketed annotations above, we have used Serre duality and universal coefficients to compute these cohomology groups.)

Finally, $H^1(\mathcal{O}^*)$ represents the isomorphism classes of line bundles, and a line bundle can be described by constant transition functions exactly when it is in the image of $H^1(\mathbb{C}^*)$. The exact sequence (1) shows that this image is the kernel of the homomorphism $H^1(\mathcal{O}^*) \to \mathbb{Z}$, which is the first Chern class.

Exercise

Both $\mathbb{CP}^1$ itself ($g = 0$) and punctured Riemann surfaces of higher genus admit projective structures with reducible holonomy. How does the proof above break down in these cases?

Notes

This proof that a $\mathbb{CP}^1$ structure with reducible holonomy has developing map which omits the holonomy fixed point is taken from [AGF30, pp. 297–300, 305–306]. Gunning presents a different argument in [Gun67, Prop. A4], based on the rank-2 vector over X arising from lifting the holonomy map to $\mathrm{SL}_2\mathbb{C}$ and acting on $\mathbb{C}^2$.

The relation between the Chern class and constant transition functions is detailed in [Gun66, Sec. 8a]. This is also a standard consequence of Chern-Weil theory, since bundles with constant transition functions admit a flat connection and the de Rham class of c_1 is represented by the trace of the curvature of a connection (see e.g. [GH78, pp. 139–144]).

References

[AGF30] P. Appell, É. Goursat, and P. Fatou. *Théorie des Fonctions Algébriques, Tome 2: Fonctions Automorphes.* Gauthier-Villars, Paris, 1930.

[GH78] P. Griffiths and J. Harris. *Principles of algebraic geometry.* Wiley-Interscience, New York, 1978. Pure and Applied Mathematics.

[Gun66] R. Gunning. *Lectures on Riemann surfaces.* Princeton Mathematical Notes. Princeton University Press, Princeton, N.J., 1966.

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