Math 535 - Complex Analysis - Spring 2010

Final Exam Solutions

(1) (a) What does it mean for a region $\Omega \subset \mathbb{C}$ to be simply connected?

Ahlfors' definition: A region Ω is simply connected if its complement in the Riemann sphere, $\hat{\mathbb{C}} \setminus \Omega$, is connected.

(Other accepted answers:

- *$n(\gamma, a) = 0$ for all closed curves $\gamma \subset \Omega$ and $a \notin \Omega$*
- *Any correct statement of the usual topological definition of "simply connected", e.g. all closed curves in Ω are null-homotopic.)*

(b) State Cauchy's theorem for analytic functions on a simply connected region Ω.

Theorem: If f is analytic on a simply connected region Ω and if $\gamma \subset \Omega$ is a cycle, then $\int_\gamma f(z)\,dz = 0$.

(Also acceptable if stated for a closed curve rather than a general cycle.)

(2) Does there exist an analytic function f on the unit disk such that

$$f(1/n) = f(-1/n) = \sin(1/n)$$

for all integers $n \geq 2$? (Either construct such a function, or prove that no such function exists.)

Suppose that f is such a function. Since $\{1/n \mid n \geq 2\}$ accumulates at 0, the condition $f(1/n) = \sin(1/n)$ for $n \geq 2$ gives $f(z) = \sin(z)$ for all $z \in \Delta$. But $\sin(-1/n) \neq \sin(1/n)$, contradicting the condition $f(-1/n) = \sin(1/n)$. Therefore no such function exists.

(3) (a) What does it mean for two regions to be conformally equivalent?

The regions Ω and Ω' are conformally equivalent if there is an analytic homeomorphism of Ω onto Ω'.

(b) Prove that the upper half-plane $\mathbb{H} = \{z \mid \mathrm{Im}(z) > 0\}$ is not conformally equivalent to the complex plane $\mathbb{C}$.

Suppose $f : \mathbb{C} \to \mathbb{H}$ is a conformal equivalence. Then f is an entire function with positive imaginary part, hence f is constant, a contradiction.

(4) Suppose that $\Omega \subset \mathbb{C}$ is a region and $\mathcal{F} \subset \mathcal{O}(\Omega)$ is a normal family of analytic functions on Ω. Define

$$\sqrt{\mathcal{F}} = \{f \in \mathcal{O}(\Omega) \mid f^2 \in \mathcal{F}\}.$$

Show that $\sqrt{\mathcal{F}}$ is a normal family.

Let $\{f_n\} \subset \sqrt{\mathcal{F}}$ be a sequence. Then $\{f_n^2\} \subset \mathcal{F}$, so there is a subsequence $\{f_{n_k}^2\}$ that is either locally uniformly convergent or which tends to ∞ locally uniformly.

If $\{f_{n_k}^2\}$ is locally uniformly convergent, then $\{|f_{n_k}|^2\}$ is locally uniformly bounded, and so $\{|f_{n_k}|\}$ is also locally uniformly bounded (since $|f|^2 < M$ implies $|f| < \sqrt{M}$). Thus $\{f_{n_k}\}$ has a locally uniformly convergent subsequence.

If $\{f_{n_k}^2\}$ tends locally uniformly to ∞, then $\{f_{n_k}\}$ also tends locally uniformly to ∞ (since $|f|^2 > M$ implies $|f| > \sqrt{M}$).

Thus any sequence in $\sqrt{\mathcal{F}}$ has a subsequence that either converges locally uniformly or which tends locally uniformly to ∞, so $\sqrt{\mathcal{F}}$ is normal.

(5) Find a polynomial $p(x, y)$ of degree 5 that is a harmonic function, and then find its harmonic conjugate.

The harmonic conjugate of

$$\mathrm{Re}(z^5) = x^5 - 10x^3y^2 + 5xy^4$$

is

$$\mathrm{Im}(z^5) = 5x^4y - 10x^2y^3 + y^5.$$

(6) Let $A = \{z|1 < |z| < 535\}$. Prove that there does not exist an analytic function $f \in \mathcal{O}(A)$ such that $\exp(f(z)) = z$ for all $z \in A$.

Suppose f is such a function. Differentiating $\exp(f(z)) = z$ gives $f'(z) = 1/z$, so dz/z is an exact 1-form on A and its integral over any cycle is zero. However, by Cauchy's theorem,

$$\int_{|z|=100} \frac{dz}{z} = 2\pi i \neq 0,$$

a contradiction.