Math 535 - Complex Analysis - Spring 2010

Final Exam

Advice: *All of these problems have relatively "short" solutions. If you find yourself writing pages of complicated calculations, you might want to stop and reconsider your approach.*

(1) (a) What does it mean for a region $\Omega \subset \mathbb{C}$ to be *simply connected*? (Give a definition.)
(b) State Cauchy's theorem for analytic functions on a simply connected region Ω.

(2) Does there exist an analytic function f on the unit disk such that

$$f(1/n) = f(-1/n) = \sin(1/n)$$

for all integers $n \geq 2$? (Either construct such a function or prove that no such function exists.)

(3) (a) What does it mean for two regions to be *conformally equivalent*? (Give a definition.)
(b) Prove that the upper half-plane $\mathbb{H} = \{z \mid \operatorname{Im}(z) > 0\}$ is *not* conformally equivalent to the complex plane $\mathbb{C}$.

(4) Suppose that $\Omega \subset \mathbb{C}$ is a region and $\mathcal{F} \subset \mathcal{O}(\Omega)$ is a normal family of analytic functions on Ω. Define

$$\sqrt{\mathcal{F}} = \{f \in \mathcal{O}(\Omega) \mid f^2 \in \mathcal{F}\}.$$

Show that $\sqrt{\mathcal{F}}$ is a normal family.

(5) Find a polynomial $p(x, y)$ of degree 5 that is a *harmonic* function, and then find its harmonic conjugate.

(6) Let $A = \{z \mid 1 < |z| < 535\}$. Prove that there does *not* exist an analytic function $f \in \mathcal{O}(A)$ such that $\exp(f(z)) = z$ for all $z \in A$.