Math 535 - Complex Analysis

Challenge Problems

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(C1) Show that any zero of the Riemann zeta function with nonzero imaginary part lies on the line $\mathrm{Re}(z) = \frac{1}{2}$.

(C2) Circles and stereographic projection.

(a) Show that stereographic projection determines a bijection between the set of circles on the unit sphere in $\mathbb{R}^3$ and the set of *generalized circles* in $\mathbb{C}$. (For the purposes of this problem, define a circle on the unit sphere to be the intersection of the sphere with a plane, whenever that intersection is nonempty and is not a single point. A generalized circle in $\mathbb{C}$ is a circle or a straight line.)

(b) Let C be a circle in $\mathbb{C}$ with center z and radius r. Find a formula for a, b, c, d such that the corresponding circle on the unit sphere is the intersection of S^2 with $\{(x, y, z) \mid ax + by + cz = d\}$.

(C3) Let $\mathcal{H}(\mathbb{C})$ denote the vector space of *entire* functions, i.e. analytic functions defined on the whole complex plane $\mathbb{C}$. Note that $\mathcal{H}(\mathbb{C})$ is closed under multiplication and composition of functions. Let $E_0 \subset \mathcal{H}(\mathbb{C})$ denote the subspace spanned by the polynomials and the exponential function $\exp(z) = e^z$, and let E denote the space of functions obtained from those in E_0 by a finite sequence of addition, multiplication, and composition. (Thus for example E contains $f(z) = 3\exp(z^2 + z\exp(ze^z)) - z^3e^{-8z}$.)

Show that the vector space $\mathcal{H}(\mathbb{C})/E$ is infinite-dimensional.

(C4) Let $f(z)$ be an analytic function defined on $\mathbb{C}$. The *Newton map* of $f(z)$ is the function

$$N_f(z) = z - \frac{f(z)}{f'(z)}.$$

Newton's method is a computational method for locating zeros of the function $f(z)$, starting from an initial guess z_0. We define a sequence $\{z_i\}$ by $z_{i+1} = N_f(z_i)$. We say that the method *succeeds* if the sequence z_i converges to a point z_∞ such that $f(z_\infty) = 0$, and that it *fails* otherwise. (Note that the method may fail because $N_f(z_k) = \infty$ for some k, in which case the iteration stops.)

(a) If $p(z) = (z - A)(z - B)$ is a quadratic polynomial with distinct roots $A, B \in \mathbb{C}$, show that Newton's method succeeds if and only if z_0 is closer to one of the two roots; equivalently, the method fails if and only if z_0 lies on the line perpendicularly bisecting the segment AB. (It might be instructive to think about the polynomials $z^2 - 1$ and $z^2 + 1$.)

(b) Say something nontrivial about what happens when Newton's method fails for a quadratic polynomial. (e.g. Why does it fail? Does the sequence always terminate at ∞ after finitely many steps, or can something else happen?)

(C5) A smooth function $f(\theta)$ on the unit circle can be extended to a harmonic function $F(z)$ on the unit disk using the Poisson integral formula. Given $f(\theta)$, define another function $g(\theta)$ on the circle by

$$g(\theta) = \frac{1}{2\pi}\int_0^{2\pi} f(\phi)\cot\left(\frac{\theta - \phi}{2}\right) d\phi.$$

Note that the integrand has a singularity at $\theta = \phi$, so the integral should be understood as a principal value

$$g(\theta) = \lim_{\epsilon \to 0^+} \frac{1}{2\pi} \int_{|\phi - \theta| > \epsilon} f(\phi) \cot\left(\frac{\theta - \phi}{2}\right) d\phi.$$

Let $G(z)$ denote the harmonic extension of $g(\theta)$. Show that $F(z)$ and $G(z)$ are harmonic conjugates, i.e. that $F(z) + iG(z)$ is an analytic function.

(C6) The identity

$$\cos(n\theta) + i\sin(n\theta) = e^{in\theta} = \left(e^{i\theta}\right)^n = (\cos(\theta) + i\sin(\theta))^n$$

shows that for each integer $n \geq 0$ there is a polynomial $T_n(x)$ with the property that

$$T_n(\cos(\theta)) = \cos(n\theta).$$

(a) Explain this. (That is, why does the existence of $T_n(x)$ follow from the identity?)
(b) Write out $T_n(x)$ for $n = 1, 2, 3, 4, 5$.
(c) Show that $T_n(x)$ is the coefficient of t^n when the function $\frac{1-xt}{1-2xt+t^2}$ is expanded as a Taylor series in t about the point $t = 0$, i.e.

$$\frac{1 - xt}{1 - 2xt + t^2} = \sum_{n=0}^{\infty} T_n(x) t^n.$$

(C7) Show that one can detect vertices of a regular n-gon in $\mathbb{C}$ with finitely many polynomial conditions; that is, for any $n \geq 3$, find a set of polynomials $F_1, F_2, \ldots, F_k$ in n variables such that the complex numbers $a_1, a_2, \ldots, a_n$ are the vertices of a regular n-gon if and only if $F_i(a_1, \ldots, a_n) = 0$ for $i = 1, 2, \ldots, k$.

Note that in your homework you showed that this is possible for $n = 3$ and $k = 1$, where $F_1(a, b, c) = a^2 + b^2 + c^2 - bc - ac - ab$.

Hint: What is the most general polynomial of degree n whose roots form a regular n-gon?

(C8) Construct a sequence of analytic functions $f_n(z)$ on a domain Ω that converge pointwise to a function $f(z)$ that is *not* analytic. (Note: It should be a bit surprising that this is possible.)

(C9) Give an example of an analytic function defined by a power series $f(z) = \sum a_n z^n$ with radius of convergence $R = 1$ such that $\sum a_n = S$ but $\lim_{z \to 1} f(z)$ does not exist. (Compare to Abel's theorem, which says that $f(z) \to S$ as z approaches 1 non-tangentially.)

(C10) Give an example of an open set Ω and an analytic function $f(z)$ on Ω that cannot be extended to an analytic function in a neighborhood of any point in $\partial\Omega$. More precisely, show that for any $z_0 \in \partial\Omega$ it is impossible to find an analytic function $g(z)$ defined in a neighborhood U of z_0 such that $g(z) = f(z)$ on $U \cap \Omega$.

(C10$\frac{1}{2}$) Give an example of an analytic function $f(z)$ on the open unit disk Δ that cannot be extended to an analytic function in a neighborhood of any point in $\partial\Delta$. More precisely, show that for any $z_0 \in \partial\Delta$ it is impossible to find an analytic function $g(z)$ defined in a neighborhood U of z_0 such that $g(z) = f(z)$ on $U \cap \Delta$.

(C11) Suppose $A_n \in \mathrm{PSL}_2(\mathbb{C})$ is a sequence of Möbius transformations such that $\|A_n\| \to \infty$ as $n \to \infty$. (Here we use the notation $\left\| \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right\| = \sqrt{|a|^2 + |b|^2 + |c|^2 + |d|^2}$.)

Show that there exists a subsequence A_{n_i} and two points $x, y \in \hat{\mathbb{C}}$ such that for any closed disk D that does not contain y, the sequence of functions $A_{n_i}(z)$ converges uniformly to the constant function x on D. (For example, if $A_n(z) = nz$, then no subsequence is necessary, and one can take $y = 0$ and $x = \infty$.)

(C12) Fix a point $p \in \mathbb{C}$ and let $\tilde{\mathscr{O}}_p$ denote the set of pairs (U, f) where U is an open neighborhood of p and f is an analytic function defined on U. We say that (U, f) and (V, g) are *equivalent as germs* if $f(z) = g(z)$ for all $z \in U \cap V$; in this case we write $(U, f) \sim (V, g)$.

(a) Show that the set of equivalence classes $\mathscr{O}_p = \tilde{\mathscr{O}}_p / \sim$ forms a ring, i.e. that pointwise addition and multiplication of functions descend to well-defined operations on the set of equivalence classes.

(b) Show that $\mathscr{O}_p$ has a unique maximal ideal, and that this ideal is generated by the equivalence class of the function $f(z) = (z - p)$.

(C13) Suppose $f(z)$ and $g(z)$ are entire functions such that $|f(z)| < |g(z)|$ for all z. Show that f is a constant multiple of g.

(C14) Let $f(z)$ be an analytic function with $f'(0) \neq 0$.

(a) Show that there is a unique Möbius transformation $A(z)$ satisfying

$$A(0) = f(0)$$
$$A'(0) = f'(0)$$
$$A''(0) = f''(0).$$

(b) Let $g(z) = A^{-1}(f(z))$. Show that $g(0) = 0$, $g'(0) = 1$, and $g''(0) = 0$. Calculate $g'''(0)$.

(C15) Develop a Poisson integral formula for extending a piecewise continuous function on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ to a harmonic function on the domain $\frac{x^2}{a^2} + \frac{y^2}{b^2} < 1$. (Use the parameterization $\gamma(t) = a\cos(t) + b\sin(t)$.)

(C16) Let $C \subset [0, 1]$ denote the standard middle-third Cantor set. Suppose that f is a bounded analytic function on $\Omega \setminus C$, where Ω is a region containing $[0, 1]$. Show that f extends to an analytic function on Ω.

(C17) Suppose $A(x)$ is a polynomial. Show that the power series

$$f(z) = \sum_{n \geq 0} A(n) z^n$$

converges on $|z| < 1$ to a *rational function* of z.

(C18) Let f be an analytic function on a region Ω with $f'(z) \neq 0$ for all $z \in \Omega$. The *Schwarzian derivative* of f is the function

$$S_f(z) = \frac{f'''(z)}{f'(z)} - \frac{3}{2}\left(\frac{f''(z)}{f'(z)}\right)^2.$$

(a) The *nonlinearity* of f is the function $N_f(z) = \frac{f''(z)}{f'(z)}$. Show that $S_f(z) = {N_f}'(z) - \frac{1}{2}N_f(z)^2$.

(b) Show that $S_f(z) \equiv 0$ if and only if f is the restriction of a Möbius transformation to Ω.

(c) Suppose A is a Möbius transformation. Show that $S_{A \circ f}(z) = S_f(z)$.