Math 442 / David Dumas / Fall 2010

Midterm Solutions

(1) Prove that if two regular surfaces intersect at only one point, then they have the same tangent plane at that point. (That is, if $S_1 \cap S_2 = \{p\}$ then $T_pS_1 = T_pS_2$.)

Solution. It is enough to show that if two regular surfaces S_1, S_2 intersect at p and $T_pS_1 \neq T_pS_2$, then the set $S_1 \cap S_2$ contains more than one point. In fact we will show that $S_1 \cap S_2$ contains a curve through p.

For $i = 1, 2$, represent S_i in a neighborhood of p as $\{(x, y, z) \mid f_i(x, y, z) = 0\}$ where f_i is a differentiable function with 0 as a regular value (so in particular $\nabla f_i(p) \neq 0$).

The tangent plane of S_i at p is the plane through p with normal vector $\nabla f_i(p)$. Since the tangent planes are different, the vectors $\nabla f_1(p)$ and $\nabla f_2(p)$ are linearly independent.

Let $f_3(x, y, z)$ be a differentiable function defined in a neighborhood of p such that $f_3(p) = 0$ and $\{\nabla f_1(p), \nabla f_2(p), \nabla f_3(p)\}$ is a basis of $\mathbb{R}^3$. (For example, complete the linearly independent set $\{\nabla f_1(p), \nabla f_2(p)\}$ to a basis by adding a vector v, and then let $f_3(q) = (p - q) \cdot v$.)

Define $F(x, y, z) = (f_1(x, y, z), f_2(x, y, z), f_3(x, y, z))$. Then $F(p) = 0$ and the rows of dF_p are linearly independent, so by the inverse function theorem F is a diffeomorphism from a neighborhood of p to a neighborhood of $(0, 0, 0)$. Let G denote the inverse of this map. Then for all $t \in \mathbb{R}$ with $|t|$ sufficiently small, $G(0, 0, t)$ is defined and lies on both S_1 and S_2 (since $f_i(G(0, 0, t)) = 0$ for $i = 1, 2$). Since G is a diffeomorphism, all of the points obtained this way are distinct. This shows that $S_1 \cap S_2$ is infinite.

Comment. The intuition behind this solution is the following: Two distinct planes through a point p in $\mathbb{R}^3$ intersect in a line. Locally, regular surfaces are well-approximated like their tangent planes, so if $T_pS_1 \neq T_pS_2$, then $S_1 \cap S_2$ is approximately a line. In fact, $S_1 \cap S_2$ is a regular curve near p whose tangent line at p is $T_pS_1 \cap T_pS_2$.

(2) Determine the set of all positive real numbers A such that the equation

$$(x + y + z)^3 = A\left(x^3 + y^3 + z^3\right)$$

defines a regular surface in $\mathbb{R}^3 - \{(0, 0, 0)\}$.

Solution. Let $F(x, y, z) = (x + y + z)^3 - A(x^3 + y^3 + z^3)$. We want to know when $F^{-1}(0)$ is a regular surface in $\mathbb{R}^3 - \{(0, 0, 0)\}$. Note that F is symmetric in x, y, and z. We first determine the critical points of F. We have

$$\frac{\partial F}{\partial x} = 3\left[(x + y + z)^2 - Ax^2\right]$$
$$\frac{\partial F}{\partial y} = 3\left[(x + y + z)^2 - Ay^2\right]$$
$$\frac{\partial F}{\partial z} = 3\left[(x + y + z)^2 - Az^2\right]$$

so critical points are defined by $x^2 = y^2 = z^2 = \frac{1}{A}(x + y + z)^2$. In particular, x, y, and z are equal up to sign at any critical point.

Consider the case $x = y = z = s$. In order for this to be a critical point we must have $As^2 = (3s)^2$, so when $A \neq 9$, the only critical point on this line is $(0,0,0)$. If however $A = 9$, then every point on the line $x = y = z$ is critical.

Now consider $x = y = -z = s$. In order for this to be a critical point we must have $As^2 = (s + s - s)^2 = s^2$, so when $A \neq 1$ the only critical point on this line is $(0,0,0)$. If however $A = 1$, then every point on the line $x = y = -z$ is critical.

By symmetry we get a similar conclusion for the cases $x = -y = z$ and $-x = y = z$, and to summarize:

- If $A \notin \{1, 9\}$, then F has no critical points other than $(0,0,0)$.
- If $A = 9$, then the critical set of F is the line $x = y = z$.
- If $A = 1$, then the critical set of F is the union of the three lines $x = y = -z$, $x = -y = z$, $-x = y = z$.

We immediately conclude that for $A \notin \{1, 9\}$, zero is a regular value of $F(x, y, z)$ on $\mathbb{R}^3 - \{(0,0,0)\}$ and $F = 0$ defines a regular surface. It remains to analyze the cases $A = 1$ and $A = 9$ separately.

Case $A = 1$: We have

$$\begin{aligned} F(x,y,z) &= (x+y+z)^3 - x^3 - y^3 - z^3 \\ &= 3x^2y + 3xy^2 + 3x^2z + 3y^2z + 3xz^2 + 3yz^2 + 6xyz \\ &= 3(x+y)(x+z)(y+z) \end{aligned}$$

So $F^{-1}(0)$ is the union of three distinct planes that meet at $(0,0,0)$. This is not a regular surface, because near a line of intersection of two of these planes (say, in an arbitrarily small neighborhood of $(1,-1,0)$) the set does not project injectively onto any of the coordinate planes.

Case $A = 9$: Suppose $S = F^{-1}(0)$ were a regular surface in $\mathbb{R}^3 - \{(0,0,0)\}$. In this case $F(s,s,s) = (3s)^3 - 9(3s^3) = 0$ so the entire line $\ell = \{(x,y,z) \mid x = y = z\}$ is contained in S. Therefore at any point $p \in \ell$, the tangent plane T_pS must contain ℓ. The cyclic permutation $(x,y,z) \mapsto (y,z,x)$ rotates $\mathbb{R}^3$ around ℓ by angle $2\pi/3$, but this permutation does not affect the value of F so it preserves S and fixes every point of ℓ. Therefore, the tangent plane to S at $p \in \ell$ must be invariant under this rotation.

Since no plane in $\mathbb{R}^3$ containing ℓ is invariant under rotation by $2\pi/3$ around ℓ, this is a contradiction, and S is not a regular surface.

Summary. The equation $(x+y+z)^3 = A(x^3+y^3+z^3)$ defines a regular surface in $\mathbb{R}^3 - \{(0,0,0)\}$ for all real numbers A except $A = 1$ and $A = 9$.

(3) (a) Define the torsion function τ of a space curve.

Solution. Parameterize the curve by arc length and let $t(s) = \alpha'(s)$, $n(s) = t'(s)/|t'(s)|$, and $b(s) = t(s) \wedge n(s)$. Then the torsion $\tau(s)$ is the real-valued function such that $b'(s) = \tau(s)n(s)$ for all s.

(b) Let $\alpha : I \to \mathbb{R}^3$ denote a regular parameterized space curve without inflection points. Show that $\alpha(I)$ lies in a plane if and only if the torsion of α is identically zero.

Solution. If the curve lies in a plane P, then all of its derivatives are parallel to that plane. Therefore $t(s)$ and $n(s)$ are parallel to P, hence they span it, and $b(s)$ is a unit normal vector to P. There are two such unit normals, but by continuity,

$b(s)$ can only assume one of these values. So $b(s)$ is a constant function, and $b'(s) = 0$. This gives $\tau(s) = 0$.

Conversely, suppose the torsion is identically zero. Then $b(s)$ is a constant function; let N denote its value. Then for all s, we have $t(s) \cdot N = n(s) \cdot N = 0$. Consider the real-valued function $f(s) = (\alpha(s) - \alpha(s_0)) \cdot N$. Then $f(s_0) = 0$ and using $t(s) \cdot N = 0$ gives $f'(s) = 0$. Therefore the function $f(s)$ is identically zero, which shows that α is contained in the plane $\{p \mid (p - \alpha(s_0)) \cdot N = 0\}$.

(4) (a) Define the curvature function κ of a plane curve.

Solution. Parameterize the curve by arc length and let $t(s) = \alpha'(s)$. Define $n(s)$ to be the unit vector such that $t(s) \cdot n(s) = 0$ and so that the ordered basis $(t(s), n(s))$ is positively oriented. Then the curvature $\kappa(s)$ is the real-valued function such that $t'(s) = \kappa(s) n(s)$.

(b) Determine the curvature function of the *cycloid*

$$\alpha(t) = (at - b\sin(t), a - b\cos(t))$$

where $a, b \in \mathbb{R}$ are constants and $a \neq 0$.

Solution. Note that the given curve is not parameterized by arc length. Up to sign the curvature is given by

$$\frac{|\alpha'(t) \wedge \alpha''(t)|}{|\alpha'(t)|^3}.$$

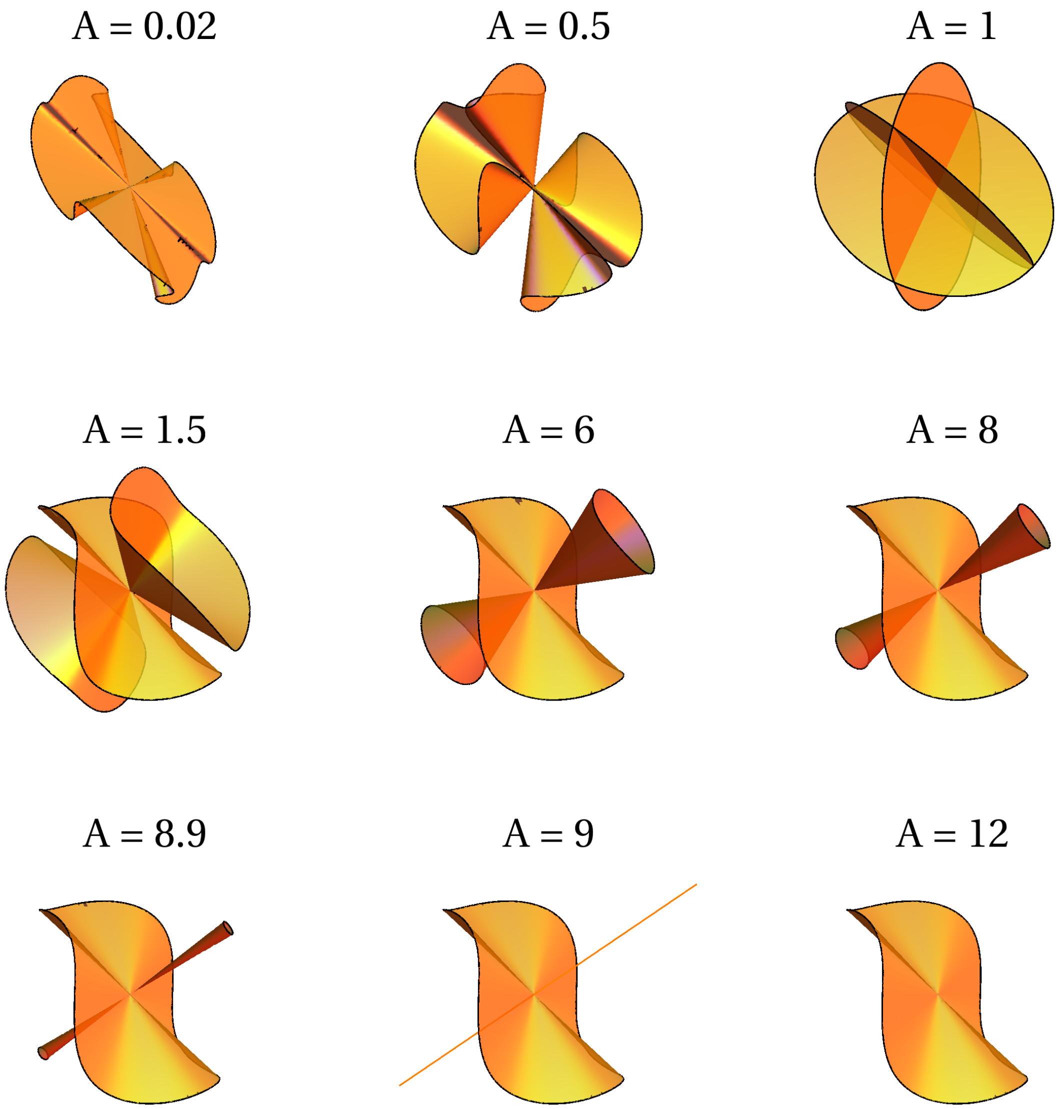
The correct sign (taking into account the definition of curvature for a plane curve) is given by replacing the numerator with $\det \begin{pmatrix} \alpha'(t) \\ \alpha''(t) \end{pmatrix}$. We calculate:

$$\begin{aligned}
\alpha'(t) &= (a - b\cos(t), b\sin(t)) \\
\alpha''(t) &= (b\sin(t), b\cos(t)) \\
|\alpha'(t)|^2 &= a^2 + b^2 - 2ab\cos(t) \\
\det \begin{pmatrix} \alpha'(t) \\ \alpha''(t) \end{pmatrix} &= b(a\cos(t) - b)
\end{aligned}$$

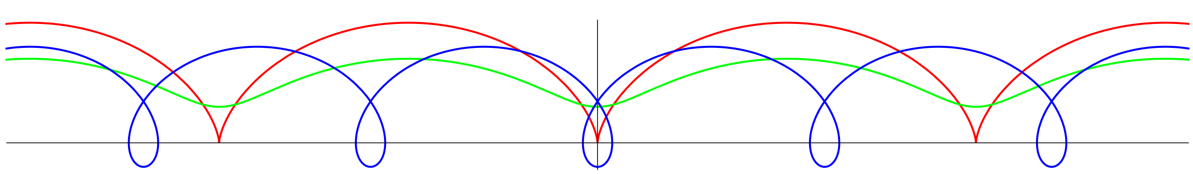
and therefore

$$\kappa(t) = \frac{b(a\cos(t) - b)}{(a^2 + b^2 - 2ab\cos(t))^{3/2}}.$$

Note that if $a = b$, the curvature is not defined for $t \in 2\pi\mathbb{Z}$.



The surface $(x+y+z)^3 = A(x^3+y^3+z^3)$ for several values of A.



Several cycloids.