Math 442 - Differential Geometry of Curves and Surfaces

Midterm Topic Outline

Version 1.0, 6th October 2010

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Note: There is no guarantee that this outline is exhaustive, though I have tried to include all of the topics we discussed. In preparing for the midterm, you should also study your notes and the assigned reading.

(1) **Curves (local theory)**

(a) A parameterized curve $\alpha : I \to \mathbb{R}^n$ is *regular* if $|\alpha'(t)| \neq 0$ for all $t \in I$. (We mostly consider n=2,3.)
(b) Review of basic notions from multivariable calculus:
  (i) Differentiability for vector-valued functions
  (ii) Arc length of a parameterized curve
  (iii) Existence of parameterization by arc length
(c) The *vector product* of u and v is the vector $u \wedge v$ such that $\langle u \wedge v, w \rangle = det(u\, v\, w)$.
(d) The *Frenet frame* of a curve $\alpha(s)$ parameterized by arc length is the triple (t, n, b) where
  (i) The *(unit) tangent vector* is $t(s) = \alpha'(s)/|\alpha'(s)|$
  (ii) The *(unit) normal vector* is $n(s) = t'(s)/|t'(s)|$
  (iii) The *(unit) binormal vector* is $b(s) = t(s) \wedge n(s)$
(e) This frame obeys the *Frenet equations*

$$\frac{d}{dt}\begin{pmatrix} t \\ n \\ b \end{pmatrix} = \begin{pmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & -\tau \\ 0 & \tau & 0 \end{pmatrix}\begin{pmatrix} t \\ n \\ b \end{pmatrix}$$

where $\kappa(s) = |\alpha'(s)|$ is the *curvature* and $\tau = \pm|n'(s)|$ is the *torsion*.
(f) Special case: If $\alpha : I \to \mathbb{R}^2$ is a plane curve, we modify the definitions slightly.
  (i) The unit normal $n(s)$ is the vector orthogonal to $t(s)$ such that (t, n) is a positive frame for $\mathbb{R}^2$.
  (ii) The (signed) curvature is the real number $\kappa(s)$ such that $t'(s) = \kappa(s)n(s)$.
(g) The *fundamental theorem of the local theory of space curves*: For any pair of functions κ, τ with $\kappa > 0$ there is a parameterized curve $\alpha : I \to \mathbb{R}^3$ with curvature κ and torsion τ. Furthermore, the resulting curve is unique up to an isometry of $\mathbb{R}^3$, i.e. if α and β have the same curvature and torsion and if the curvature is everywhere positive, then

$$\alpha(s) = A \cdot \beta(s) + v$$

for some orthogonal matrix A and vector $v \in \mathbb{R}^3$.
(h) The fundamental theorem follows from the isometry invariance of κ, τ and the existence and uniqueness of solutions to ODE with a given initial condition.
(i) *Osculation*
  (i) The *osculating plane* is $\mathrm{span}(t(s), n(s))$.
  (ii) The *osculating circle* is the circle in the osculating plane with radius $1/\kappa(s)$ centered at $\alpha(s) + (1/\kappa(s))n(s)$. It is tangent to the curve at $\alpha(s)$ and it has the same curvature as α at that point.
(j) A curve is planar if and only if $\tau(s) \equiv 0$.
(k) A planar curve is a circle if and only if κ is constant and nonzero.

(2) **Plane curves**

(a) *Crofton's formula and integral geometry.*

(i) The *space of lines* in the plane (denoted $\mathscr{L}$) can be parameterized by pairs (p, θ) where p is the orthogonal distance from a line to $(0, 0)$ and θ is the angular coordinate of the point realizing this distance.

(ii) Formally, $\mathscr{L}$ is the quotient of $\mathbb{R}^2$ by the equivalence relation generated by
- $(p, \theta) \sim (-p, \theta + \pi)$ for all $p, \theta \in \mathbb{R}$.
- $(0, \theta) \sim (0, \theta')$ for all $\theta, \theta' \in \mathbb{R}$.

(iii) *Natural measure.* An isometry of $\mathbb{R}^2$ takes lines to lines, and thus induces a map $\mathscr{L} \to \mathscr{L}$. The measure $dpd\theta$ is invariant under these maps.

(iv) If C is a regular curve in $\mathbb{R}^2$, let $N_C(p, \theta)$ denote the number of points of intersection of C with the (p, θ)-line (when this intersection is finite).

(v) *Crofton's formula:* If C is a regular curve of length ℓ then

$$\iint_{\mathscr{L}} N_C(p, \theta)\, dpd\theta = 2\ell$$

(vi) Part of the Crofton theorem is that the function N_C is integrable, e.g. that lines intersecting C in infinitely many points account for a set of zero $dpd\theta$-measure.

(vii) If $\Omega \subset \mathbb{R}^2$ is an open set bounded by a finite union of regular closed curves, let $m_\Omega(p, \theta)$ denote the total length of the intervals of intersection of Ω and the (p, θ) line.

(viii) Generalized Crofton formula: If a set Ω as above has area A, then

$$\iint_{\mathscr{L}} m_\Omega(p, \theta) dpd\theta = \pi A$$

(b) The isoperimetric inequality

(i) Let Ω be an open set in $\mathbb{R}^2$ bounded by a closed regular curve C, where Ω has area A and C has length L. Then

$$L^2 \geq 4\pi A.$$

Furthermore, if $L^2 = 4\pi A$ then C is a circle.

(ii) Corollary: The circle minimizes perimeter among curves enclosing a fixed area.

(iii) Corollary: The circle maximizes enclosed area among curves with a fixed length.

(iv) One proof of the isoperimetric inequality uses Crofton's formula to show that the integral of a certain positive real-valued function on $\mathscr{L} \times \mathscr{L}$ is a positive multiple of $L^2 - 4\pi A$.

(3) **Surfaces**

(a) Definition of a regular surface: A subset $S \in \mathbb{R}^3$ such that for each $p \in S$ there is a neighborhood V in $\mathbb{R}^3$ and a map $X : U \to V \cap S$, where $U \subset \mathbb{R}^2$ is open, satisfying:

(i) X is differentiable

(ii) X is a homeomorphism

(iii) X is an immersion, i.e. for each $q \in U$, the differential dX_q is injective.

(b) Equivalent definitions: Locally, a regular surface is

(i) The graph of a differentiable function over one of the coordinate planes xy, xz, or yz.

(ii) The graph of a differentiable function over *some* plane in $\mathbb{R}^3$.

(iii) The inverse image of a regular value of a differentiable function $F(x, y, z)$.

(iv) The image of the xy plane under a diffeomorphism from an open set in $\mathbb{R}^3$ to $\mathbb{R}^3$.

(c) Parameterizations: A differentiable map $X : U \to \mathbb{R}^3$ with injective differential at every point is a *immersion* or a *regular parameterized surface*; after restricting to a sufficiently small open set $V \subset U$, the image is a regular surface. In other words, the image of an immersion is locally regular.

(d) The *inverse function theorem*: If $\phi : U \to V$ is a differentiable map and if $d\phi_p$ is an isomorphism ($\Leftrightarrow$ the matrix of partial derivatives at p is invertible), then ϕ is a diffeomorphism *near* p, i.e. there exists a neighborhood U' of p such that $\phi : U' \to V' = \phi(U')$ is a diffeomorphism.

(e) General philosophy: Many ideas from multivariable calculus can be generalized to regular surfaces. Often the generalization is defined like this: Use local coordinates to move everything into $\mathbb{R}^2$, then apply the usual definition for functions of two variables.

(f) *Differentiable functions on surfaces.*

(i) A function $f : S \to \mathbb{R}$ on a regular surface can be locally expressed as $f(u, v)$, where (u, v) are local coordinates on S near a point $p = (u_0, v_0)$.

(ii) If $f(u, v)$ is differentiable (in the multivariable calculus sense) at (u_0, v_0), then we say f is *differentiable at* p.

(iii) This definition does not depend on the coordinate system, since a change of coordinates is differentiable.

(iv) If f is differentiable at every point of S, then it is *differentiable.*

(g) *Differentiable maps between surfaces.*

(i) A continuous map $\phi : S_1 \to S_2$ between two regular surfaces can be locally expressed as $f(u, v) = (s(u, v), t(u, v))$, where (u, v) are local coordinates on S_1 near $p = (u_0, v_0)$ and (s, t) are local coordinates on S_2 near $\phi(p)$.

(ii) If $s(u, v)$ and $t(u, v)$ are differentiable at p, then we say ϕ is *differentiable* at p.

(iii) If ϕ is differentiable at every point of S_1, then ϕ is *differentiable.*

(h) *Tangent plane.* If $X(u, v)$ is a local parameterization of S, then the span of X_u and X_v at a point p is the *tangent plane of S at p*, denoted T_pS.

(i) An alternate definition of the tangent plane: Consider the set of all curves in S that pass through p. The set consisting of their tangent vectors at p is T_pS.

(j) *Differential.* A differentiable map $\phi : S_1 \to S_2$ induces a linear map $d\phi_p : T_pS_1 \to T_{\phi(p)}S_2$, the *differential of ϕ at p.* In local coordinates (u, v) near p and (s, t) near $\phi(p)$, we can write $\phi(u, v) = (s(u, v), t(u, v))$. Then the differential has matrix

$$d\phi_p = \begin{pmatrix} \frac{\partial s}{\partial u}(p) & \frac{\partial s}{\partial v}(p) \\ \frac{\partial t}{\partial u}(p) & \frac{\partial t}{\partial v}(p) \end{pmatrix}.$$

(k) The *inverse function theorem for surfaces.* If $\phi : S_1 \to S_2$ is a differentiable map and $d\phi_p$ is an isomorphism, then ϕ is a diffeomorphism *near* p, i.e. there exists a neighborhood U' of p such that $\phi : U' \to V' = \phi(U')$ is a diffeomorphism.

(l) A map $\phi : S_1 \to S_2$ whose differential is an isomorphism at every point need not be injective or surjective.
Examples:
- The inclusion of a small disk by a coordinate chart (not surjective).
- The plane mapping to the torus by a doubly-periodic parameterization function (not injective).

(m) Some examples of regular surfaces:
- A graph $z = f(x, y)$.
- Inverse image of a regular value $\{(x, y, z) \mid F(x, y, z) = c\}$.
- *Surface of revolution.* Rotate a plane curve $\beta(t)$ around a line, use t and rotation angle θ as parameters.

- The surface of revolution of a circle that does not intersect the axis is a *circular torus*.
- A surface that contains a line segment through each of its point is *ruled*. Such a surface can be parameterized by $X(s,t) = \alpha(s) + t\beta(s)$ where α is a space curve and β is a nonzero vector-valued function.
- *Surface of tangents.* $X(s,t) = \alpha(s) + t\alpha'(s)$ where α is a space curve parameterized by arc length. This surface is ruled.
- *Tubes.* Let $X(s,\theta) = \alpha(s) + \epsilon\cos(\theta)n(s) + \epsilon\sin(\theta)b(s)$, where α is a space curve with unit normal n and unit tangent t, and $\epsilon > 0$ is the tube radius.
- The *cone* on the space curve $\alpha(t)$ is parameterized by $X(s,t) = t\alpha(s)$.

(n) Some examples of diffeomorphisms:

- If $S \subset \mathbb{R}^3$ is a regular surface and $F : \mathbb{R}^3 \to \mathbb{R}^3$ is a diffeomorphism of $\mathbb{R}^3$ that preserves S, i.e. $F(S) = S$, then the restriction of F is a diffeomorphism $F : S \to S$.
- A surface of revolution has a natural family of diffeomorphisms R_θ obtained by rotating the surface by angle θ around its axis of symmetry.
- The map $(x,y,0) \mapsto (x,y,f(x,y))$ from a coordinate plane to the graph of a differentiable function is a diffeomorphism.
- A local parameterization $X : U \to S$ of a regular surface is a diffeomorphism from U to $X(U)$.