Formula Reference Sheet

- Curvature and torsion for an arbitrary parameterization $\alpha(t)$:

$$\kappa(t) = \frac{|\alpha'(t) \wedge \alpha''(t)|}{|\alpha'(t)|^3}$$

$$\tau(t) = -\frac{\langle \alpha'(t) \wedge \alpha''(t), \alpha'''(t) \rangle}{|\alpha'(t) \wedge \alpha''(t)|^2}$$

- Index notation and (u, v)-notation for:

  First fundamental form (metric) $\begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix} = \begin{pmatrix} E & F \\ F & G \end{pmatrix}$

  Inverse metric $\begin{pmatrix} g^{11} & g^{12} \\ g^{21} & g^{22} \end{pmatrix} = \begin{pmatrix} E & F \\ F & G \end{pmatrix}^{-1}$

  Second fundamental form $\begin{pmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{pmatrix} = \begin{pmatrix} e & f \\ f & g \end{pmatrix}$

- Definition of the Christoffel symbols Γ^i_{jk} and the second fundamental form L_{jk} for a parameterization $X(u_1, u_2)$ with oriented normal N:

$$\frac{\partial^2 X}{\partial u_j \partial u_k} = \sum_i \Gamma^i_{jk} \frac{\partial X}{\partial u_i} + L_{jk} N$$

- General formula for the Christoffel symbols in terms of the metric:

$$\Gamma^i_{jk} = \frac{1}{2} \sum_\ell g^{i\ell} \left(\frac{\partial g_{j\ell}}{\partial u_k} + \frac{\partial g_{\ell k}}{\partial u_j} - \frac{\partial g_{jk}}{\partial u_\ell} \right)$$

- Formula for the Christoffel symbols in an orthogonal parameterization (where $g_{12} = g_{21} = 0$, $g_{11} = E$, $g_{22} = G$):

$$\Gamma^1 = \begin{pmatrix} \Gamma^1_{11} & \Gamma^1_{12} \\ \Gamma^1_{21} & \Gamma^1_{22} \end{pmatrix} = \frac{1}{2E} \begin{pmatrix} E_u & E_v \\ E_v & -G_u \end{pmatrix}$$

$$\Gamma^2 = \begin{pmatrix} \Gamma^2_{11} & \Gamma^2_{12} \\ \Gamma^2_{21} & \Gamma^2_{22} \end{pmatrix} = \frac{1}{2G} \begin{pmatrix} -E_v & G_u \\ G_u & G_v \end{pmatrix}$$

- Gaussian and mean curvature in terms of the first and second fundamental forms:

$$K = \frac{eg - f^2}{EG - F^2} \qquad H = \frac{eG + gE - 2fF}{2(EG - F^2)}$$

- Intrinsic formula for Gaussian curvature in an orthogonal parameterization:

$$K = -\frac{1}{2\sqrt{EG}} \left[\frac{\partial}{\partial v} \left(\frac{E_v}{\sqrt{EG}} \right) + \frac{\partial}{\partial u} \left(\frac{G_u}{\sqrt{EG}} \right) \right]$$

- The covariant derivative of a vector field $w(t) = (w_1(t), w_2(t))$ along a curve $\alpha(t) = (u_1(t), u_2(t))$:

$$\left(\frac{Dw}{dt} \right)_i = \frac{dw_i}{dt} + \sum_{j,k} \Gamma^i_{jk} w_j \frac{du_k}{dt}$$

- The geodesic equations:

$$u_i'' + \sum_{j,k} \Gamma^i_{jk} u_j' u_k' = 0$$