Math 442 - Differential Geometry of Curves and Surfaces

Final Exam Topic Outline

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Note: There is no guarantee that this outline is exhaustive, though I have tried to include all of the topics we discussed. In preparing for the final exam, you should also study your notes and the assigned reading.

(1) **Curves (local theory)**

(a) A parameterized curve $\alpha : I \to \mathbb{R}^n$ is *regular* if $|\alpha'(t)| \neq 0$ for all $t \in I$. (We mostly consider n=2,3.)
(b) Review of basic notions from multivariable calculus:
  (i) Differentiability for vector-valued functions
  (ii) Arc length of a parameterized curve
  (iii) Existence of parameterization by arc length
(c) The *vector product* of u and v is the vector $u \wedge v$ such that $\langle u \wedge v, w \rangle = det(u\, v\, w)$.
(d) The *Frenet frame* of a curve $\alpha(s)$ parameterized by arc length is the triple (t, n, b) where
  (i) The *(unit) tangent vector* is $t(s) = \alpha'(s)/|\alpha'(s)|$
  (ii) The *(unit) normal vector* is $n(s) = t'(s)/|t'(s)|$
  (iii) The *(unit) binormal vector* is $b(s) = t(s) \wedge n(s)$
(e) This frame obeys the *Frenet equations*

$$\frac{d}{dt}\begin{pmatrix} t \\ n \\ b \end{pmatrix} = \begin{pmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & -\tau \\ 0 & \tau & 0 \end{pmatrix}\begin{pmatrix} t \\ n \\ b \end{pmatrix}$$

where $\kappa(s) = |\alpha'(s)|$ is the *curvature* and $\tau = \pm|n'(s)|$ is the *torsion.*
(f) Special case: If $\alpha : I \to \mathbb{R}^2$ is a plane curve, we modify the definitions slightly.
  (i) The unit normal $n(s)$ is the vector orthogonal to $t(s)$ such that (t, n) is a positive frame for $\mathbb{R}^2$.
  (ii) The (signed) curvature is the real number $\kappa(s)$ such that $t'(s) = \kappa(s)n(s)$.
(g) The *fundamental theorem of the local theory of space curves*: For any pair of functions κ, τ with $\kappa > 0$ there is a parameterized curve $\alpha : I \to \mathbb{R}^3$ with curvature κ and torsion τ. Furthermore, the resulting curve is unique up to an isometry of $\mathbb{R}^3$, i.e. if α and β have the same curvature and torsion and if the curvature is everywhere positive, then

$$\alpha(s) = A \cdot \beta(s) + v$$

for some orthogonal matrix A and vector $v \in \mathbb{R}^3$.
(h) The fundamental theorem follows from the isometry invariance of κ, τ and the existence and uniqueness of solutions to ODE with a given initial condition.
(i) *Osculation*
  (i) The *osculating plane* is $\text{span}(t(s), n(s))$.
  (ii) The *osculating circle* is the circle in the osculating plane with radius $1/\kappa(s)$ centered at $\alpha(s) + (1/\kappa(s))n(s)$. It is tangent to the curve at $\alpha(s)$ and it has the same curvature as α at that point.
(j) A curve is planar if and only if $\tau(s) \equiv 0$.
(k) A planar curve is a circle if and only if κ is constant and nonzero.

(2) **Plane curves**

(a) *Crofton's formula and integral geometry.*

(i) The *space of lines* in the plane (denoted $\mathscr{L}$) can be parameterized by pairs (p, θ) where p is the orthogonal distance from a line to $(0, 0)$ and θ is the angular coordinate of the point realizing this distance.

(ii) Formally, $\mathscr{L}$ is the quotient of $\mathbb{R}^2$ by the equivalence relation generated by

- $(p, \theta) \sim (-p, \theta + \pi)$ for all $p, \theta \in \mathbb{R}$.
- $(0, \theta) \sim (0, \theta')$ for all $\theta, \theta' \in \mathbb{R}$.

(iii) *Natural measure.* An isometry of $\mathbb{R}^2$ takes lines to lines, and thus induces a map $\mathscr{L} \to \mathscr{L}$. The measure $dpd\theta$ is invariant under these maps.

(iv) If C is a regular curve in $\mathbb{R}^2$, let $N_C(p, \theta)$ denote the number of points of intersection of C with the (p, θ)-line (when this intersection is finite).

(v) *Crofton's formula:* If C is a regular curve of length ℓ then

$$\iint_{\mathscr{L}} N_C(p, \theta)\, dpd\theta = 2\ell$$

(vi) Part of the Crofton theorem is that the function N_C is integrable, e.g. that lines intersecting C in infinitely many points account for a set of zero $dpd\theta$-measure.

(vii) If $\Omega \subset \mathbb{R}^2$ is an open set bounded by a finite union of regular closed curves, let $m_\Omega(p, \theta)$ denote the total length of the intervals of intersection of Ω and the (p, θ) line.

(viii) Generalized Crofton formula: If a set Ω as above has area A, then

$$\iint_{\mathscr{L}} m_\Omega(p, \theta) dpd\theta = \pi A$$

(b) The isoperimetric inequality

(i) Let Ω be an open set in $\mathbb{R}^2$ bounded by a closed regular curve C, where Ω has area A and C has length L. Then

$$L^2 \geq 4\pi A.$$

Furthermore, if $L^2 = 4\pi A$ then C is a circle.

(ii) Corollary: The circle minimizes perimeter among curves enclosing a fixed area.

(iii) Corollary: The circle maximizes enclosed area among curves with a fixed length.

(iv) One proof of the isoperimetric inequality uses Crofton's formula to show that the integral of a certain positive real-valued function on $\mathscr{L} \times \mathscr{L}$ is a positive multiple of $L^2 - 4\pi A$.

(3) **Surfaces**

(a) Definition of a regular surface: A subset $S \in \mathbb{R}^3$ such that for each $p \in S$ there is a neighborhood V in $\mathbb{R}^3$ and a map $X : U \to V \cap S$, where $U \subset \mathbb{R}^2$ is open, satisfying:

(i) X is differentiable

(ii) X is a homeomorphism

(iii) X is an immersion, i.e. for each $q \in U$, the differential dX_q is injective.

(b) Equivalent definitions: Locally, a regular surface is

(i) The graph of a differentiable function over one of the coordinate planes xy, xz, or yz.

(ii) The graph of a differentiable function over *some* plane in $\mathbb{R}^3$.

(iii) The inverse image of a regular value of a differentiable function $F(x, y, z)$.

(iv) The image of the xy plane under a diffeomorphism from an open set in $\mathbb{R}^3$ to $\mathbb{R}^3$.

(c) Parameterizations: A differentiable map $X : U \to \mathbb{R}^3$ with injective differential at every point is a *immersion* or a *regular parameterized surface*; after restricting to a sufficiently small open set $V \subset U$, the image is a regular surface. In other words, the image of an immersion is locally regular.

(d) The *inverse function theorem*: If $\phi : U \to V$ is a differentiable map and if $d\phi_p$ is an isomorphism ($\Leftrightarrow$ the matrix of partial derivatives at p is invertible), then ϕ is a diffeomorphism *near* p, i.e. there exists a neighborhood U' of p such that $\phi : U' \to V' = \phi(U')$ is a diffeomorphism.

(e) General philosophy: Many ideas from multivariable calculus can be generalized to regular surfaces. Often the generalization is defined like this: Use local coordinates to move everything into $\mathbb{R}^2$, then apply the usual definition for functions of two variables.

(f) *Differentiable functions on surfaces.*
   (i) A function $f : S \to \mathbb{R}$ on a regular surface can be locally expressed as $f(u, v)$, where (u, v) are local coordinates on S near a point $p = (u_0, v_0)$.
   (ii) If $f(u, v)$ is differentiable (in the multivariable calculus sense) at (u_0, v_0), then we say f is *differentiable at* p.
   (iii) This definition does not depend on the coordinate system, since a change of coordinates is differentiable.
   (iv) If f is differentiable at every point of S, then it is *differentiable.*

(g) *Differentiable maps between surfaces.*
   (i) A continuous map $\phi : S_1 \to S_2$ between two regular surfaces can be locally expressed as $f(u, v) = (s(u, v), t(u, v))$, where (u, v) are local coordinates on S_1 near $p = (u_0, v_0)$ and (s, t) are local coordinates on S_2 near $\phi(p)$.
   (ii) If $s(u, v)$ and $t(u, v)$ are differentiable at p, then we say ϕ is *differentiable* at p.
   (iii) If ϕ is differentiable at every point of S_1, then ϕ is *differentiable.*

(h) *Tangent plane.* If $X(u, v)$ is a local parameterization of S, then the span of X_u and X_v at a point p is the *tangent plane of* S *at* p, denoted T_pS.

(i) An alternate definition of the tangent plane: Consider the set of all curves in S that pass through p. The set consisting of their tangent vectors at p is T_pS.

(j) *Differential.* A differentiable map $\phi : S_1 \to S_2$ induces a linear map $d\phi_p : T_pS_1 \to T_{\phi(p)}S_2$, the *differential of* ϕ *at* p. In local coordinates (u, v) near p and (s, t) near $\phi(p)$, we can write $\phi(u, v) = (s(u, v), t(u, v))$. Then the differential has matrix

$$d\phi_p = \begin{pmatrix} \frac{\partial s}{\partial u}(p) & \frac{\partial s}{\partial v}(p) \\ \frac{\partial t}{\partial u}(p) & \frac{\partial t}{\partial v}(p) \end{pmatrix}.$$

(k) The *inverse function theorem for surfaces.* If $\phi : S_1 \to S_2$ is a differentiable map and $d\phi_p$ is an isomorphism, then ϕ is a diffeomorphism *near* p, i.e. there exists a neighborhood U' of p such that $\phi : U' \to V' = \phi(U')$ is a diffeomorphism.

(l) A map $\phi : S_1 \to S_2$ whose differential is an isomorphism at every point need not be injective or surjective.
   Examples:
   - The inclusion of a small disk by a coordinate chart (not surjective).
   - The plane mapping to the torus by a doubly-periodic parameterization function (not injective).

(m) Some examples of regular surfaces:
   - A graph $z = f(x, y)$.
   - Inverse image of a regular value $\{(x, y, z) \mid F(x, y, z) = c\}$.
   - *Surface of revolution.* Rotate a plane curve $\beta(t)$ around a line, use t and rotation angle θ as parameters.

- The surface of revolution of a circle that does not intersect the axis is a *circular torus*.
- A surface that contains a line segment through each of its point is *ruled*. Such a surface can be parameterized by $X(s,t) = \alpha(s) + t\beta(s)$ where α is a space curve and β is a nonzero vector-valued function.
- *Surface of tangents.* $X(s,t) = \alpha(s) + t\alpha'(s)$ where α is a space curve parameterized by arc length. This surface is ruled and α is the line of striction (see section 3.5).
- *Surface of binormals.* $X(s,t) = \alpha(s) + tb(s)$ where α is a space curve parameterized by arc length and $b(s)$ is the unit binormal vector. This surface is ruled and the curve α is a geodesic (we did not prove this, but see exercise 17 in section 4.4).
- *Tubes.* Let $X(s,\theta) = \alpha(s) + \epsilon\cos(\theta)n(s) + \epsilon\sin(\theta)b(s)$, where α is a space curve with unit normal n and unit tangent t, and $\epsilon > 0$ is the tube radius.
- The *cone* on the space curve $\alpha(t)$ is parameterized by $X(s,t) = t\alpha(s)$.

(n) Some examples of diffeomorphisms:
- If $S \subset \mathbb{R}^3$ is a regular surface and $F : \mathbb{R}^3 \to \mathbb{R}^3$ is a diffeomorphism of $\mathbb{R}^3$ that preserves S, i.e. $F(S) = S$, then the restriction of F is a diffeomorphism $F : S \to S$.
- A surface of revolution has a natural family of diffeomorphisms R_θ obtained by rotating the surface by angle θ around its axis of symmetry.
- The map $(x, y, 0) \mapsto (x, y, f(x, y))$ from a coordinate plane to the graph of a differentiable function is a diffeomorphism.
- A local parameterization $X : U \to S$ of a regular surface is a diffeomorphism from U to $X(U)$.

(4) **Geometry of surfaces: First fundamental form, length, angle, and area**

(a) Restricting the inner product of $\mathbb{R}^3$ makes T_pS into an *inner product space*. The associated quadratic form is the *first fundamental form*, denoted I_p. Thus $I_p(w)$ is the squared length of w (as a vector in $\mathbb{R}^3$).

(b) In the basis X_u, X_v for T_pS given by a local parameterization, the matrix of I_p is $\begin{pmatrix} E & F \\ F & G \end{pmatrix}$ where $E = \langle X_u, X_u\rangle$ $F = \langle X_u, X_v\rangle$ $G = \langle X_v, X_v\rangle$.
In other words, we have "$I = \langle dX, dX\rangle$".

(c) The length of a curve $\alpha(t) = (u(t), v(t))$ on S is given by

$$\int_a^b \sqrt{Eu'^2 + 2Fu'v' + Gv'^2}dt.$$

Note that Eu'^2 means $E(u(t), v(t))\,(u'(t))^2$, and similarly for the other terms.

(d) The area of a region Ω contained in a local coordinate chart (u, v) is given by

$$\iint_\Omega \sqrt{EG - F^2}dudv.$$

Note that when S is contained in $\mathbb{R}^2$, this is the usual formula for change of variables, and $\sqrt{EG - F^2}$ is the Jacobian of the transformation.

(e) The angle θ between vectors $w_1 = aX_u + bX_v$ and $w_2 = cX_u + dX_v$ satisfies

$$\cos(\theta) = \frac{\langle w_1, w_2\rangle}{|w_1|\,|w_2|} = \frac{Eac + F(ad + bc) + Gbd}{\sqrt{(Ea^2 + 2Fab + Gb^2)(Ec^2 + 2Fcd + Gd^2)}}$$

(f) A map between surfaces whose differential preserves length of vectors is a (local) *isometry*.

(5) **Geometry of surfaces: Gauss map, second fundamental form, and curvature**

(a) An *orientation* of a surface is a choice of a unit normal vector at each point in such a way that the resulting map $N : S \to S^2$ is continuous. Here $S^2 = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\}$. If a surface S has an orientation, then it has exactly two, and we say S is *orientable.*

(b) The map $N : S \to S^2$ associated with an orientation is called the *Gauss map* of the surface.

(c) When parameterizing an oriented surface, we always choose $X(u, v)$ so that $X_u \wedge X_v$ is a positive multiple of the unit normal, i.e.

$$N(u, v) = \frac{X_u \wedge X_v}{|X_u \wedge X_v|}$$

(d) The tangent plane to S^2 at $N(u, v)$ is naturally identified with the tangent plane to S at $X(u, v)$; in each case, the plane consists of vectors orthogonal to $N(u, v)$. In particular, the differential of the Gauss map is a linear map $dN_p : T_pS \to T_pS$.

(e) The differential of the Gauss map is self-adjoint with respect to I_p (so it is diagonalizable and has real eigenvalues with orthogonal eigenspaces).

(f) The *second fundamental form* is the quadratic form II_p on T_pS defined by $\mathit{II}_p(w) = -\langle dN_p(w), w\rangle = \langle \frac{\partial^2 X}{\partial w^2}, N(p)\rangle$. So II_p is the normal component of the acceleration of a path in S with tangent vector w. One could summarize this definition as "$\mathit{II} = -\langle dX, dN\rangle = \langle d^2X, N\rangle$".

(g) The eigenvalues of $-dN_p$ are the *principal curvatures* of S at p, denoted k_1, k_2. The associated eigenspaces are the *principal directions.*

(h) The product of the principal curvatures is the *Gaussian curvature*

$$K(p) = k_1(p)k_2(p) = \det(dN_p).$$

The average of the principal curvatures is the *mean curvature*

$$H(p) = \frac{1}{2}(k_1(p) + k_2(p)) = -\frac{1}{2}\mathrm{tr}(dN_p).$$

(i) In local coordinates, the matrix of II_p is given by $\begin{pmatrix} e & f \\ f & g \end{pmatrix}$ where:

$$\begin{aligned}
e &= \langle X_{uu}, N\rangle = -\langle X_u, N_u\rangle \\
f &= \langle X_{uv}, N\rangle = -\langle X_u, N_v\rangle = -\langle X_v, N_u\rangle \\
g &= \langle X_{vv}, N\rangle = -\langle X_v, N_v\rangle
\end{aligned}$$

(j) This is different from the matrix of $-dN_p$, unless X_u and X_v are orthonormal. In general, we have $dN_p = -\begin{pmatrix} E & F \\ F & G \end{pmatrix}^{-1} \begin{pmatrix} e & f \\ f & g \end{pmatrix}$.

(k) Using the formula for N in terms of X_u and X_v gives the convenient formula

$$e = \frac{1}{\sqrt{EG - F^2}} \det(X_u \;\; X_v \;\; X_{uu})$$

and similarly for f and g, replacing only the second derivative term with X_{uv} or X_{vv}, respectively.

(l) Using the formula for dN_p, we have

$$K = \frac{eg - f^2}{EG - F^2} \quad \text{and} \quad H = \frac{Eg - 2Ff + Gg}{2(EG - F^2)}$$

and the principal curvatures are the roots of the polynomial $\lambda^2 - 2H\lambda + K$.

(m) If $\alpha(s)$ is a curve contained in S, then the length of the projection of $\alpha''(s)$ onto $N(\alpha(s))$ is the *normal curvature* of α, denoted k_N. The normal curvature at $\alpha(s)$ only depends on $\alpha'(s)$, and is given by $II_{\alpha(s)}(\alpha'(s))$. Here we assume $\alpha(s)$ is parameterized by arc length.

(n) The principal curvatures at p are the extreme values of the normal curvature as α' varies over all unit tangent vectors at p.

(o) Classification of points on a surface:
- If $K(p) > 0$, then p is an *elliptic point.*
- If $K(p) = 0$ but dN_p is nonzero, then p is a *parabolic point.*
- If $K(p) = 0$ and dN_p is zero, then p is a *planar point.*
- If $K(p) < 0$, then p is a *hyperbolic point.*
- If $k_1(p) = k_2(p)$ (or equivalently, $H(p)^2 = K(p)$), then p is an *umbilic point.*

(p) Typical examples:
- Every point on the unit sphere is elliptic and umbilic.
- Every point on a cylinder is parabolic.
- Every point on a plane is planar
- The point $(0,0,0)$ on $\{z = (x^2+y^2)^2\}$ is planar.
- The point $(0,0,0)$ on the "saddle" $\{z^2 = x^2 - y^2\}$ is hyperbolic.
- If $f''(x) > 0$, then every point on the surface of revolution of f is hyperbolic.
- The point $(0,0,0)$ on the circular paraboloid $\{z = x^2 + y^2\}$ is umbilic.

(q) A curve in S whose tangent vector at each point is a principal direction is a *line of curvature.*

(r) Special cases:
   (i) If $F = 0$, then the horizontal and vertical lines in the uv plane correspond to orthogonal curves in S.
   (ii) If $F = f = 0$, then the principal curvatures are e/E and g/G, the principal directions are X_u and X_v, and the horizontal and vertical lines in the uv plane correspond to lines of curvature in S.

(s) *Isothermal coordinates.* For any $p \in S$ there is a local coordinate system $X(u,v)$ that is orthogonal and in which $|X_u| = |X_v| = \lambda(u,v)$. Equivalently, the first fundamental form is $E = G = \lambda^2$, $F = 0$.

(t) The Gaussian curvature in an isothermal coordinate system is given by $K = -(1/\lambda^2)\Delta\log(\lambda)$.

(6) Intrinsic geometry of surfaces

(a) The *Gauss frame* of an oriented surface with local parameterization $X(u,v)$ is the frame $(X_u, X_v, N(u,v))$ where $N(u,v) = (X_u \wedge X_v)/|X_u \wedge X_v|$.

(b) The derivative of the Gauss frame can be expressed in terms of the Gauss frame, giving

$$\begin{aligned}
X_{uu} &= \Gamma_{11}^1 X_u + \Gamma_{11}^2 X_v + eN\\
X_{uv} &= \Gamma_{12}^1 X_u + \Gamma_{12}^2 X_v + fN\\
X_{vu} &= \Gamma_{21}^1 X_u + \Gamma_{21}^2 X_v + fN\\
X_{vv} &= \Gamma_{22}^1 X_u + \Gamma_{22}^2 X_v + gN
\end{aligned}$$

where the coefficients Γ_{jk}^i are the *Christoffel symbols.*

(c) In index notation, let $L_{ij} = \begin{pmatrix} e & f \\ f & g \end{pmatrix}$ and write X_i for $\partial X/\partial u_i$ and X_{ij} for $\partial^2 X/\partial u_i \partial u_j$. Then

$$X_{ij} = \Gamma_{jk}^i X_i + L_{ij} N.$$

(d) Relations like $\langle X_{uu}, X_u\rangle = \frac{1}{2}E_u$ connect the Christoffel symbols to the first fundamental form (see (2) on p232). This leads to the formula

$$\Gamma^i_{jk} = \frac{1}{2}\sum_{\ell} g^{i\ell}\left(\frac{\partial g_{j\ell}}{\partial u_k} + \frac{\partial g_{\ell k}}{\partial u_j} - \frac{\partial g_{jk}}{\partial u_\ell}\right)$$

where $g_{ij} = \begin{pmatrix} E & F \\ F & G\end{pmatrix}$ is the first fundamental form and g^{ij} is its inverse matrix.

(e) Equality of mixed third partial derivatives (e.g. $X_{uuv} = X_{uvu}$) gives a system of nine equations relating Christoffel symbols, the first and second fundamental forms, and their derivatives. These equations are of two types:
   (i) Gauss equations: Express the Gaussian curvature K as a function of Christoffel symbols and their first derivatives.
   (ii) Codazzi-Mainardi equations: Express derivatives of second fundamental form (e.g. $e_v - f_u$) in terms of the second fundamental form and the Christoffel symbols (e.q. $e\Gamma^1_{12} + f(\Gamma^2_{12} - \Gamma^1 11) - g\Gamma^2_{11}$).

(f) The Gauss equation implies the *Theorema Egregium*: Locally isometric surfaces have the same Gaussian curvature.

(g) Bonnet's fundamental theorem: If E, F, G, e, f, g are differentiable functions and if E, F, G describe a positive definite form at each point, then these arise locally as the first and second fundamental forms of a parameterized surface if and only if they obey the Gauss and Codazzi-Mainardi equations. Furthermore, if two surfaces have the same first and second fundamental form with respect to some local coordinate system, then the surfaces are related by an isometry of $\mathbb{R}^3$.

(h) *The covariant derivative.*
   (i) Let W be a vector field on a regular surface S with local parameterization $X(u, v)$. Then the partial derivatives $\partial W/\partial u$ and $\partial W/\partial v$ may not be tangent to S. The projections of these vectors to the tangent plane are the *covariant derivatives* $DW/\partial u$ and $DW/\partial v$.
   (ii) More generally, if α is a curve in S, then the projection of $\frac{d}{dt}(W(\alpha(t))$ to $T_{\alpha(t)}S$ is the covariant derivative of W along α, denoted DW/dt.
   (iii) The covariant derivative is linear and only depends on the tangent vector to α: If $\alpha'(t) = a(t)X_u + b(t)X_v$ then

$$\frac{DW}{dt} = a\frac{DW}{\partial u} + b\frac{DW}{\partial v}.$$

   (iv) The covariant derivative DW/dt only depends on the values of a vector field on the curve itself, and is therefore defined for a *vector field along a curve.*
   (v) The tangent vectors $\alpha'(t)$ form a vector field along a curve α. Its covariant derivative is the *acceleration* $D\alpha'(t)/dt$.

(i) *Parallelism.*
   (i) A vector field W along a curve is *parallel* if $DW/dt = 0$.
   (ii) The angle between a pair of parallel vector fields is constant. More generally,

$$\frac{d}{dt}\langle W, V\rangle = \langle\frac{DW}{dt}, V\rangle + \langle W, \frac{DV}{dt}\rangle.$$

   (iii) Given a vector $W(0) \in T_{\alpha(0)}S$, there is a unique extension to a parallel vector field $W(t)$ along α.
   (iv) With $W(0)$ extended to $W(t)$ as above, the value $W(t) \in T_{\alpha(t)}S$ is called the *parallel transport* of $W(0)$ from $\alpha(0)$ to $\alpha(t)$ along α.
   (v) If two surfaces are tangent along a curve, then the parallel transport of a vector along the curve can be computed in either surface and the results will be equal.

(j) *Geodesics.*
   (i) A *geodesic* is a curve whose unit tangent vector field is parallel.

(ii) If $\alpha(s)$ is parameterized by arc length, then $D\alpha'/ds$ is orthogonal to $\alpha'(s)$. Therefore, there is a real number $k_g(s)$ such that

$$\alpha'(s) \wedge \frac{D\alpha'}{ds} = k_g(s) N(\alpha(s))$$

where N is the unit normal vector to the surface. The quantity $k_g(s)$ is the *geodesic curvature* of α. (Its sign depends on the orientation induced by the local parameterization.)

(iii) Alternate definition of a geodesic: a curve with $k_g \equiv 0$.

(iv) Given a point $p \in S$ and a unit vector $v \in T_pS$ there is a unique geodesic $\gamma : (-\epsilon, \epsilon) \to S$ such that $\gamma(0) = p$ and $\gamma'(0) = v$.

(v) In local coordinates (u_1, u_2), the geodesic equations are

$$\frac{d^2 u_i}{dt^2} + \sum_{j,k} \Gamma^i_{jk} \frac{du_j}{dt} \frac{du_k}{dt} = 0$$

(vi) Geodesic examples:
- Straight lines in $\mathbb{R}^2$
- Great circles on S^2
- Helices on a right circular cylinder

(vii) Geodesics on a surface of revolution:
- All meridians are geodesics
- A parallel is a geodesic if and only if it is a local maximum of distance to the axis.
- *Clairaut's relation.* Let $\theta(s)$ denote the angle between $\alpha'(s)$ and the parallel it intersects at $\alpha(s)$, and let $r(s)$ denote the distance from $\alpha(s)$ to the axis of revolution. Then the quantity $r(s)\cos(\theta(s))$ is constant along any geodesic.

(7) **The Gauss-Bonnet Theorem**

(a) *Regions and curves.*

(i) A *piecewise smooth curve* on a surface has a pair of tangent vectors at each of its vertices. The angle between these is an *exterior angle* or *turning angle* of the curve. The exterior angle is signed, where a positive angle means that the tangent vector turns counterclockwise about the unit normal.

(ii) A *simple region* on a surface is a closed set homeomorphic to a disk bounded by a piecewise smooth simple closed curve.

(iii) A *regular region* on a surface is a compact set whose boundary is a finite disjoint union of piecewise smooth simple closed curves.

(b) *Gauss-Bonnet v1.0.* If $R \subset S$ is a simple region with smooth boundary that is contained in a single coordinate chart of the surface, then

$$\int_{\partial R} k_g(s) ds + \iint_R K dA = 2\pi.$$

(c) For an orthogonal coordinate system, the proof of the Gauss-Bonnet theorem has two ingredients:

(i) Green's theorem

(ii) The *theorem of the turning tangents*: If $\gamma : [0, \ell] \to \mathbb{R}^2$ is a smooth simple closed curve parameterized by arc length with signed curvature κ, then

$$\int_0^\ell \kappa(s) ds = \pm 2\pi$$

where the sign indicates whether the disk bounded by γ lies to the left (+) or to the right (-) as γ is traversed in the direction of increasing s.

(d) *Gauss-Bonnet v2.0.* If $R \subset S$ is a simple region with piecewise smooth boundary that is contained in a single coordinate chart of the surface, and if $\theta_1, \dots, \theta_n$ are the external angles of the positively oriented boundary of R, then

$$\sum_{i=1}^{n} \theta_i + \int_{\partial R} k_g(s)ds + \iint_R K dA = 2\pi.$$

(e) A *triangulation* of a regular region is a decomposition into triangles (simple regions with three vertices) such that the triangles are pairwise disjoint unless they share exactly one vertex or exactly one edge.

(f) If R is a triangulated regular region with V vertices, E edges, and F triangles, then the Euler characteristic of R is the quantity

$$\chi(R) = V - E + F.$$

The Euler characteristic depends only on R and not on the particular triangulation.

(g) Any regular region has a triangulation such that each triangle is contained in a single coordinate chart.

(h) *Gauss-Bonnet v3.0.* If R is a regular region on a surface S (e.g. $R = S$ if S is compact) whose positively oriented boundary has external angles $\theta_1, \dots, \theta_n$, then we have

$$\sum_{i=1}^{n} \theta_i + \int_{\partial R} k_g(s)ds + \iint_R K dA = 2\pi\chi(R).$$

(i) *The classification of compact orientable surfaces.*
   (i) The *genus* of a compact orientable surface is a nonnegative integer that uniquely determines its homeomorphism type, i.e. two surfaces are homeomorphic if and only if they have the same genus.
   (ii) The sphere has genus $g = 0$, the torus has $g = 1$.
   (iii) Informally, the genus $g(S)$ is the number of "handles" that one must attach to the sphere in order to create a surface homeomorphic to S.
   (iv) Let $C = \cup_{i=1}^{g} C_i$ where C_i is the circle in $\mathbb{R}^2$ with center $(2i-1, 0)$ and radius 1. Let S be the set of points in $\mathbb{R}^3$ whose distance from C is exactly $1/4$. Then S is a compact (topological) surface of genus g.

(j) Corollaries of GB v3.0.
   (i) If S is a compact orientable regular surface, then

$$\iint_S K dA = 2\pi\chi(S).$$

   (ii) Gauss-Bonnet v2.1: The restriction that R must lie in a single coordinate chart can be dropped from v2.0.
   (iii) If a regular surface has positive Gaussian curvature, then it is homeomorphic to the sphere.
   (iv) If $S \subset \mathbb{R}^3$ is a compact regular surface that is not homeomorphic to the sphere, then S contains points of positive, negative, and zero Gaussian curvature.

(8) **Exponential map and normal coordinates**

(a) Fix a regular surface $S \subset \mathbb{R}^3$ and a point $p \in S$. For any $v \in T_pS$ there exists a unique constant-speed parameterized geodesic $\gamma(t)$ such that $\gamma(0) = p$ and $\gamma'(0) = v$. Write $\gamma(t) = \gamma(t, v)$ to emphasize the dependence on v.

(b) Homogeneity: $\gamma(t, \lambda v) = \gamma(\lambda t, v)$ for any $\lambda \in \mathbb{R}$. Thus for small v the geodesic is defined for a large time interval.

(c) The *exponential map* at $p \in S$ is defined by

$$\exp_p(v) = \gamma(1, v)$$

i.e. send $v \in T_pS$ to the endpoint of the geodesic segment of length $|v|$ starting at p in the direction of v.

(d) The map $\exp_p$ is defined on an open subset of T_pS that contains $(0, 0)$. Thus there exists δ such that $\exp_p$ is defined for all $v \in T_pS$ with $|v| < \delta$.

(e) The differential of $\exp_p$ at 0 is the identity, i.e.

$$(d \exp_p)_0(v) = v$$

for any $v \in T_pS \simeq T_0T_pS$.

(f) By the IFT, the exponential map is a local diffeomorphism. The diffeomorphic image of a neighborhood of $0 \in T_pS$ is a *normal neighborhood* of p.

(g) Composing orthonormal rectangular coordinates for T_pS and $\exp_p$ gives *normal coordinates* for S centered at p.

(h) Composing polar coordinates for T_pS and $\exp_p$ gives *geodesic polar coordinates* for S centered at p.

(i) In normal coordinates, $E(p) = G(p) = 1$ and $F(p) = 0$.

(j) In geodesic polar coordinates, $E \equiv 1$, $F \equiv 0$, and G satisfies

$$\lim_{\rho \to 0} G(\rho, \theta) = 0 \qquad \lim_{\rho \to 0} \frac{\partial}{\partial \rho} \sqrt{G(\rho, \theta)} = 1.$$

(Compare these to polar coordinates in $\mathbb{R}^2$.)

(k) Theorem: For all $p \in S$ there is a neighborhood U of p that is also a normal neighborhood of q for all $q \in U$.

(l) For all $p \in S$ there is a normal neighborhood U of p such that any geodesic segment in U is *minimizing*, i.e. its length is less than or equal to that of any path in S with the same endpoints.

(m) Theorem: For all $p \in S$ there exists a normal neighborhood $U \in S$ that is *convex*, i.e. for all $q_1, q_2 \in U$ there is a unique geodesic segment with endpoints q_1, q_2 that is contained in U.

(n) Corollary: Every compact surface has a finite geodesic triangulation (i.e. a triangulation with finitely many triangles, each of which is bounded by three geodesic segments).