Math 442 / Differential Geometry of Curves and Surfaces / David Dumas

Challenge Problems

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Please read the guidelines for submitting challenge problems in the course syllabus.

(C1) Describe all curves on the unit sphere with constant torsion. Are any of them closed? (Hint: Begin with the case $\tau = 0$.)

(C2) Let $\alpha : I \to \mathbb{R}^3$ be a differentiable curve parameterized by arc length, with curvature $\kappa_\alpha(s) \neq 0$ and torsion $\tau_\alpha(s)$. For each $s \in I$, let $\beta(s)$ denote the center of the osculating circle of α at $\alpha(s)$.

(a) Compute the speed $|\beta'(s)|$, curvature $\kappa_\beta(s)$, and torsion $\tau_\beta(s)$ of the curve β. (*Warning: s is not necessarily the arc length parameter for β!*)

(b) Find a particular curve α so that the new curve β is congruent to α, i.e. the two curves are related by a rotation and/or translation.

(C3) (a) Let $\alpha : I \to \mathbb{R}^3$ be a differentiable curve that lies on the unit sphere (i.e. $|\alpha(s)| = 1$ for all $s \in I$). Show that $\kappa(s) \geq 1$ for all $s \in I$.

(b) Suppose instead that α lies on the ellipsoid $ax^2 + by^2 + cz^2 = 1$, where $a, b, c > 0$. What is the minimum possible value for the curvature?

(C4) Let $\alpha : I \to R^2$ be a differentiable plane curve with positive, increasing curvature (i.e. $\kappa(s), \kappa'(s) > 0$).

(a) Show that the osculating circles of α are nested, meaning that if $s' > s$, then the osculating circle at $\alpha(s')$ is contained in the osculating circle at $\alpha(s)$.

(b) Show that it is impossible for all of the osculating circles to have a common center.

(C5) A space curve $\alpha : \mathbb{R} \to \mathbb{R}^3$ is *bounded* if there is a constant M such that $|\alpha(t)| < M$ for all $t \in \mathbb{R}$. Recall that a curve $\alpha : \mathbb{R} \to \mathbb{R}^3$ of constant curvature and constant torsion is a helix and therefore is not bounded. Given any pair of constants $K > 0$ and $T \geq 0$, show that there are bounded space curves with almost constant curvature K and almost constant torsion T. More precisely, for any $\varepsilon > 0$, find a bounded space curve $\alpha : \mathbb{R} \to \mathbb{R}^3$ such that for all t we have

$$K - \varepsilon < \kappa(t) < K + \varepsilon \quad \text{and} \quad T - \varepsilon < \tau(t) < T + \varepsilon.$$

(C6) For a differentiable curve $\alpha : I \to \mathbb{R}^n$ parameterized by arc length, the Frenet frame is defined by applying the Gram-Schmidt orthonormalization procedure to the vectors $\alpha(s), \alpha'(s), \alpha''(s), \ldots, \alpha^{(n-1)}(s)$. This frame is only defined when these n vectors are linearly independent.

(a) Show that this is indeed a generalization of the Frenet frame that we defined for a curve in $\mathbb{R}^3$, i.e. that applying the Gram-Schmidt process to $\alpha, \alpha', \alpha''$ gives t, n, b.

(b) Show that the Frenet equations for $\mathbb{R}^n$ have a coefficient matrix whose only nonzero entries are immediately above and below the diagonal. (These coefficients generalize the curvature and torsion of a curve in $\mathbb{R}^3$.)

(C7) There is a generalization of Crofton's formula of the following form: If U is a bounded open set in $\mathbb{R}^3$, then

$$\text{Volume}(U) = C \int_{\text{Planes P}\subset\mathbb{R}^3} \text{Area}(U \cap P) dP$$

where C is a constant that does not depend on U and dP is a measure on the space of all planes in $\mathbb{R}^3$.

(a) What is the dimension of the space of all planes in $\mathbb{R}^3$?

(b) Introduce a coordinate system for the space of planes in $\mathbb{R}^3$ and find an isometry-invariant measure on this space. (This generalizes the measure $dpd\theta$ on the space of lines in the plane.)

(c) By considering the open unit ball in $\mathbb{R}^3$ and the area of its intersection with a plane, calculate the constant C in this generalization of Crofton's formula.

(C8) A short arc from a very large circle is nearly a straight line segment. Similarly, one might expect that if the curvature of a plane curve is not too large, then a sufficiently short piece of the curve will be "close" to a line segment. In this problem you will work toward justifying this intuition.

Let K and ε be positive real numbers satisfying $K\varepsilon < 1$. Suppose $\alpha : \mathbb{R} \to \mathbb{R}^2$ is a plane curve parameterized by arc length whose curvature satisfies $|\kappa(s)| < K$ for all s.

(a) Let $p = \alpha(0)$ and $q = \alpha(\varepsilon)$. Show that

$$\varepsilon \cos(K\varepsilon) \leq |p - q| \leq \varepsilon.$$

(b) Let $v = (q-p)/|q-p|$ and define $\beta(t) = p + \frac{t}{\varepsilon}v$. Note that $\beta(0) = \alpha(0)$ and $\beta(\varepsilon) = \alpha(\varepsilon)$. Find an upper bound for $|\alpha(t) - \beta(t)|$, $t \in [0, \varepsilon]$, that depends only on K and ε and which tends to zero as $K\varepsilon \to 0$.

(C9) Let α be a simple closed curve in the plane. Define the *isoperimetric ratio* of α as the positive real number

$$r(\alpha) = \frac{L(\alpha)^2}{A(\alpha)}$$

where $L(\alpha)$ and $A(\alpha)$ are the length and enclosed area of α, respectively. If $\mathcal{S}$ is a collection of simple closed curves, the *optimal isoperimetric ratio* of $\mathcal{S}$ is defined as

$$R(\mathcal{S}) = \inf_{\alpha\in\mathcal{S}} r(\alpha).$$

Thus, for example, if $\mathcal{S}$ consists of all simple closed curves in the plane, then $R(\mathcal{S}) = 4\pi$ and $r(\alpha) = 4\pi$ if and only if α is a circle.

(a) Let $\mathcal{P}_n$ denote the set of all convex polygons in the plane with n sides. Determine $R(\mathcal{P}_n)$.

(b) Find all $p \in \mathcal{P}_n$ such that $r(p) = R(\mathcal{P}_n)$.

(C10) Consider the parabola $P = \{y = x^2\}$ in the plane. Let $p(s)$ denote an arc length parameterization of P with $p(0) = (0,0)$ and $p'(0) = (1,0)$. Let $T(s)$ be the tangent line to P at $p(s)$. For any $s \in \mathbb{R}$, apply a rotation and translation so that $p(s)$ is sent to $(s,0)$ and $T(s)$ becomes the x axis; call the resulting parabola $P(s)$. We say $P(s)$ is the result of *rolling P along the x axis.*

Given a point q in $\mathbb{R}^2$, we can form a path $q(s)$ by applying the same rotation and translation to q as is used to transform P into $P(s)$. One can think of q as being "rigidly attached" to P, so it moves as P rolls. Show that if $q = (0, 1/4)$, then $q(s)$ is a catenary. (If you do not know what a catenary is, then consider the problem to be "Find a simple formula for the curve traced out by $q(s)$.")

(C11) Let $\alpha(\theta)$ and $\beta(\theta)$ denote a pair of circles in $\mathbb{R}^3$ parameterized with constant speed by $\theta \in [0, 2\pi]$. Suppose that α and β lie in distinct parallel planes, and that the line joining their centers is perpendicular to these planes.

For each θ, let L_θ denote the line in $\mathbb{R}^3$ containing $\alpha(\theta)$ and $\beta(\theta)$. The union of these lines, $S = \bigcup_\theta L_\theta$ is the *scroll* generated by α and β.

Show that this scroll surface S can be defined by a simple equation $F(x, y, z) = 0$, and give a formula for F in terms of the relative position of α and β and their parameterizations.

(C12) Let $U = \{(x, y) \mid x^2 + y^2 < 1\}$ denote the unit disk in $\mathbb{R}^2$.
   (a) Let $p, q \in U$. Show that there exists is a diffeomorphism $\phi : U \to U$ such that $\phi(p) = q$.
   (b) Let $p, q \in U$ and suppose $\max(|p|, |q|) < R < 1$. Show that there exists a diffeomorphism $\phi : U \to U$ such that $\phi(p) = q$ and so that $\phi(x) = x$ for all $x \in U$ with $|x| > R$.

(C13) Let S be a connected regular surface in $\mathbb{R}^3$. Show that for any $p, q \in S$ there is a diffeomorphism $\phi : S \to S$ such that $\phi(p) = q$.
Hints:
   - *See the previous problem.*
   - *Given $p \in S$, what can you say about the set of $q \in S$ for which such a differomorphism exists?*

(C14) Characterize all ruled surfaces that are also surfaces of revolution.

(C15) Find a parameterization of the path traced out by one focus of an elliptical object as it rolls along the x axis without slipping. Your parameterization may need to use functions defined in terms of integrals that cannot be evaluated explicitly.

(C16) Show that the surface of rotation of the curve described in the previous problem has constant mean curvature.

(C17) Determine the set of pairs of real numbers (α, β) such that
$$(x + y + z)^3 = \alpha(x^3 + y^3 + z^3) + \beta(x^2y + y^2z + z^2x + x^2z + z^2y + y^2x)$$
defines a regular surface in $\mathbb{R}^3 - \{(0, 0, 0)\}$.

(C18) Find a regular surface in $\mathbb{R}^3$ that is not a sphere but which contains at least three circles through each point.

(C19) Let $S \subset \mathbb{R}^3$ be a regular surface. Let α be a line of curvature of S. Find a formula for the curvature of α (considered as a space curve) in terms of the principal curvatures of S and their covariant derivatives.

(C20) Is every space curve a line of curvature on some regular surface? (Either construct such a surface for any given curve, or give an example in which you prove that no such surface exists.)

(C21) Is every space curve a geodesic on some regular surface? (Either construct such a surface for any given curve, or give an example in which you prove that no such surface exists.)