Math 442 - Differential Geometry of Curves and Surfaces

Supplementary Homework Problems

Version 2024-06-12

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(P1) In $\mathbb{R}^3$, the vector product takes two vectors v_1, v_2 as input and produces a new vector $v_1 \wedge v_2$.

The right generalization of the vector product to $\mathbb{R}^n$ is an operation that takes $(n-1)$ vectors $v_1, v_2, \ldots, v_{n-1}$ and produces a vector $v_1 \wedge v_2 \wedge \cdots \wedge v_{n-1}$. As in the $\mathbb{R}^3$ case, the dot product of the result with another vector is expressed as a determinant.

(a) Based on these hints, write a proper definition of the vector product in $\mathbb{R}^n$.

(b) Write a formula for the components of the vector product in $\mathbb{R}^4$. (That is, write each component of $u \wedge v \wedge w$ in terms of the components of u, v, and w.)

(c) Write a similar formula for the vector product in $\mathbb{R}^2$, and describe the resulting vector geometrically. Note that this vector product takes a single vector u and produces a new vector, which you might call $\widehat{u}$.

(P2) This problem continues the discussion of matrix-valued functions from the lecture on January 21.

(a) Give an example of a differentiable matrix-valued function $M(s)$ satisfying both of these conditions:

(i) For all s, the matrix $M(s)$ is orthogonal and $\det M(s) = 1$. (That is, $M(s) \in \mathrm{SO}(3)$.)

(ii) The three columns of $M(s)$ do not form the Frenet frame of any differentiable space curve.

(b) Consider the determinant of an $n \times n$ matrix as a function of the n^2 entries of the matrix. Compute its gradient, and use this to compute

$$\frac{d}{ds} \det A(s) \bigg|_{s=0}$$

where $A(s)$ is a differentiable matrix-valued function, and $A(0) = I$.

(A formula was stated in class. Here you are asked to derive it yourself.)

(P3) For each of the following subsets of $\mathbb{R}^2$, draw a picture of the corresponding set of lines (in the (p, θ) plane) that intersect the set. For example, as we discussed in class, a point in $\mathbb{R}^2$ gives rise to a vertical sinusoidal curve in the space of lines. Shade your drawing according to the number of intersections.

(a) The unit circle.

(b) The L^1 unit circle, i.e. $\{(x, y) \mid |x| + |y| = 1\}$.

(c) The elliptic curve $\{(x, y) \mid y^2 = x^3 + 3x^2\}$, which is pictured in Figure **??**. In this case, also indicate the set of tangent lines in your drawing.

(P4) Let Ω be a convex set in $\mathbb{R}^2$. The *average chord length* of Ω is defined by

$$\bar{\ell}_\Omega = \frac{\int_{\mathscr{L}} \text{Length}(\Omega \cap (p, \theta)\text{-line})\, dp d\theta}{\int_{\mathscr{L}} dp d\theta}.$$

Find a formula for $\bar{\ell}_\Omega$ in terms of the area and perimeter of Ω.

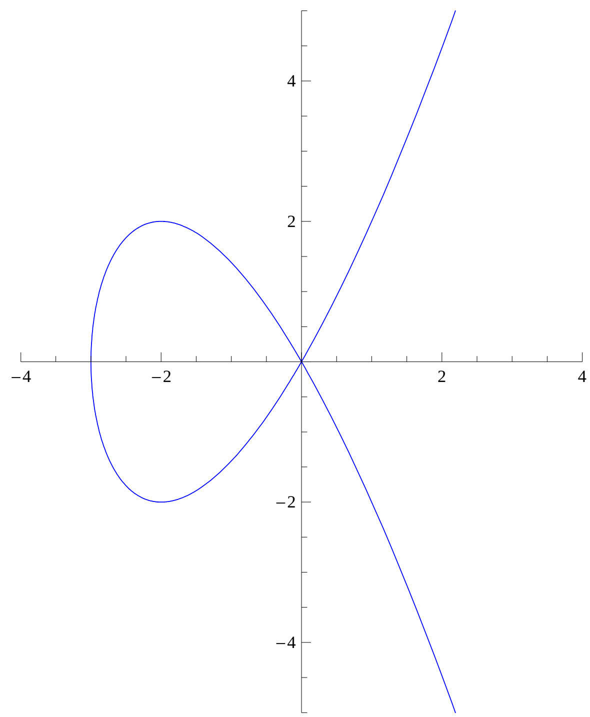

FIGURE 1. The elliptic curve $\{(x, y) \mid y^2 = x^3 + 3x^2\}$.

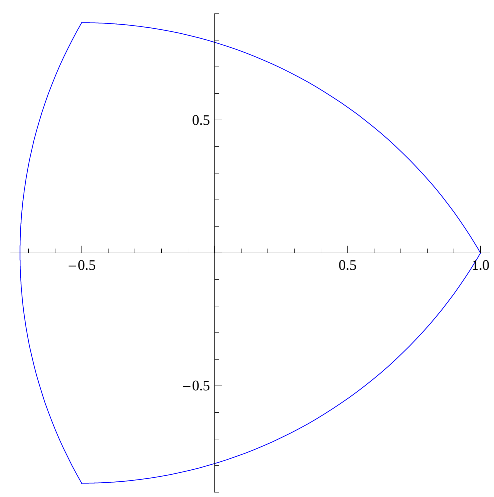

FIGURE 2. The Reuleaux triangle, which is not a triangle, consists of three circular arcs. Each arc is centered at one vertex of an equilateral triangle, and has two sides of the triangle as radii.

(P5) The *horizontal width* of a convex set $\Omega \subset \mathbb{R}^2$ is the difference between the largest and smallest x-coordinates of points in Ω. The θ*-width* of Ω is the horizontal width of the set obtained by rotating Ω clockwise by angle θ.

A *region of constant width* w is a convex set in $\mathbb{R}^2$ such that the θ-width is equal to w for all θ. A round disk is an example of such a region, as is the region bounded by the Reuleaux triangle (Figure **??**).

Show that the length of the boundary curve of a region of constant width w is equal to πw.

(P6) (a) Write the definition of a *regular curve in* $\mathbb{R}^3$ that would correspond to the definition of a regular surface on page 52 of the textbook. Your definition should begin: "A subset $C \subset \mathbb{R}^3$ is a regular curve if..."

(b) Is the following statement true or false? (Give a proof or counterexample.) *If $I \subset \mathbb{R}$ is an open interval and $\alpha : I \to \mathbb{R}^3$ is a differentiable map such that $|\alpha'(t)| \neq 0$ for all $s \in I$, then the trace of α is a regular curve.*

(P7) Consider the equations

$$(\diamondsuit) \quad x^2 + y^2 + z^2 = a^2$$

$$(\clubsuit) \quad y - 2by + z^2 = 0$$

(a) For what values of a does ($\diamondsuit$) determine a regular surface?

(b) For what values of b does ($\clubsuit$) determine a regular surface?

(c) For what values of (a, b) does the pair of equations define a regular curve? Draw an example when the intersection of the surfaces is not regular.

(d)-(f) Answer the same questions for the following pair of equations:

$$(\heartsuit) \quad xyz = a^2$$

$$(\spadesuit) \quad x^2 + y^2 - z^2 = b^2$$

(P8) As discussed in lecture, if u, v are functions of t, and if $L(u, v, u', v', t)$ is a differentiable function of 5 variables, then the functional

$$\mathscr{L}[u, v] = \int_a^b L(u(t), v(t), u'(t), v'(t), t)dt$$

has two associated Euler-Lagrange equations:

$$\frac{\partial L}{\partial u} = \frac{d}{dt}\left(\frac{\partial L}{\partial u'}\right)$$

$$\frac{\partial L}{\partial v} = \frac{d}{dt}\left(\frac{\partial L}{\partial v'}\right)$$

This is a system of second-order ODE in u, v.

(a) Given a first fundamental form $\begin{pmatrix} E & F \\ F & G \end{pmatrix}$ on an open set U, consider the length functional

$$\text{Length}(u(t), v(t)) = \int_a^b \sqrt{E(u(t), v(t))\ u'(t)^2 + 2F(u(t), v(t))\ u'(t)v'(t) + G(u(t), v(t))\ v'(t)^2}dt.$$

Write out the corresponding Euler-Lagrange equations, and simplify as much as possible.

(b) Let $U = \{(x, y) \mid y > 0\}$ be the open upper half-plane, and let $E(x, y) = G(x, y) = 1/y^2$, $F(x, y) = 0$. Substitute this into the equations from (a) and simplify. (Your answer should be a system of equations for $x(t)$ and $y(t)$.)

(c) Verify that $x(t) = \sinh(t)/\cosh(t)$, $y(t) = 1/\cosh(t)$ is a solution to the equations from (b). What does this curve look like?

Note: this part of the problem was corrected on March 2; the previous version had x and y switched, so the curve did not lie in the upper half-plane!

(P9) Prove that locally, any curve in the plane is a flow line of *some* vector field.

Here is a more precise version of what you should prove: Let $\alpha(t)$ be a regular parameterized curve in $\mathbb{R}^2$, defined for $t \in (a, b)$. Show that for any $t_0 \in (a, b)$, there is an open interval I with $t_0 \in I \subset (a, b)$ and a vector field W defined on an open neighborhood of $\alpha(I)$ such that the restriction of α to I is a flow line of W.

(P10) Let S be the surface in $\mathbb{R}^3$ defined by the equation $z^2 = x^2 + y^2 + 1$. Let C_h be the circle of intersection of S and the plane $z = h$, where $h > 1$, and let $\alpha(t)$ be a parameterization of C_h. Compute the angle between $\alpha'(0)$ and its parallel transport around C_h. Is there any h for which C_h a geodesic?

Hint: Use the same method that the book describes for the unit sphere in $\mathbb{R}^3$.

(P11) Let S be a circular torus of revolution in $\mathbb{R}^3$, parameterized by $(u, v) \in [0, 2\pi] \times [0, 2\pi]$ so that $u = 0$ is the circle of points farthest from the axis of revolution. Let θ denote the angle between a tangent vector in $T_{(0,0)}S$ and the curve $\{u = 0\}$. For each value of θ, draw a picture of what you think the geodesic ray starting at $(0, 0)$ in the direction θ might look like. Also draw a picture of what this ray looks like in the square $[0, 2\pi] \times [0, 2\pi]$ in the uv plane.

(a) $\theta = 0$ (easy!)
(b) $\theta = 0.1$
(c) $\theta = \pi/4$
(d) $\theta = \pi/2 - 0.1$
(e) $\theta = \pi/2$ (easy!)

Note: Your answer will be considered "correct" if it appears to be consistent with Clairaut's theorem and your previous homework problems about geodesics on circular tori.