Math 442 - Differential Geometry of Curves and Surfaces
Final Exam Topic Outline
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Note: There is no guarantee that this outline is exhaustive, though I have tried to include all of the topics we discussed. In preparing for the final exam, you should also study your notes and the assigned reading.

(1) **Curves (local theory)**

   (a) A parameterized curve $\alpha : I \to \mathbb{R}^n$ is *regular* if $|\alpha'(t)| \neq 0$ for all $t \in I$. (We mostly consider n=2,3.)
   (b) Review of basic notions from multivariable calculus:
      (i) Differentiability for vector-valued functions
      (ii) Arc length of a parameterized curve
      (iii) Existence of parameterization by arc length
   (c) The *vector product* of u and v is the vector $u \wedge v$ such that $\langle u \wedge v, w \rangle = det(u\, v\, w)$.
   (d) The *Frenet frame* of a curve $\alpha(s)$ parameterized by arc length is the triple (t, n, b) where
      (i) The *(unit) tangent vector* is $t(s) = \alpha'(s)/|\alpha'(s)|$
      (ii) The *(unit) normal vector* is $n(s) = t'(s)/|t'(s)|$
      (iii) The *(unit) binormal vector* is $b(s) = t(s) \wedge n(s)$
   (e) This frame obeys the *Frenet equations*

$$\frac{d}{dt} \begin{pmatrix} t \\ n \\ b \end{pmatrix} = \begin{pmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & -\tau \\ 0 & \tau & 0 \end{pmatrix} \begin{pmatrix} t \\ n \\ b \end{pmatrix}$$

   where $\kappa(s) = |\alpha'(s)|$ is the *curvature* and $\tau = |n'(s)|$ is the *torsion*.
   (f) Special case: If $\alpha : I \to \mathbb{R}^2$ is a plane curve, we modify the definitions slightly.
      (i) The unit normal $n(s)$ is the vector orthogonal to $t(s)$ such that (t, n) is a positive frame for $\mathbb{R}^2$.
      (ii) The (signed) curvature is the real number $\kappa(s)$ such that $t'(s) = \kappa(s)n(s)$.
   (g) The *fundamental theorem of the local theory of space curves*: For any pair of functions κ, τ with $\kappa > 0$ there is a parameterized curve $\alpha : I \to \mathbb{R}^3$ with curvature κ and torsion τ. Furthermore, the resulting curve is unique up to an isometry of $\mathbb{R}^3$, i.e. if α and β have the same curvature and torsion and if the curvature is everywhere positive, then

$$\alpha(s) = A \cdot \beta(s) + v$$

   for some orthogonal matrix A and vector $v \in \mathbb{R}^3$.
   (h) The fundamental theorem follows from the isometry invariance of κ, τ and the existence and uniqueness of solutions to ODE with a given initial condition.
   (i) *Osculation*
      (i) The *osculating plane* is span$(t(s), n(s))$.
      (ii) The *osculating circle* is the circle in the osculating plane with radius $1/\kappa(s)$ centered at $\alpha(s) + (1/\kappa(s))n(s)$. It is tangent to the curve at $\alpha(s)$ and it has the same curvature as α at that point.
   (j) A curve is planar if and only if $\tau(s) \equiv 0$.
   (k) A planar curve is a circle if and only if κ is constant and nonzero.

(2) **Plane curves**

   (a) *Crofton's formula and integral geometry.*
      (i) The *space of lines* in the plane (denoted $\mathscr{L}$) can be parameterized by pairs (p, θ) where p is the orthogonal distance from a line to $(0, 0)$ and θ is the angular coordinate of the point realizing this distance.
     (ii) Formally, $\mathscr{L}$ is the quotient of $\mathbb{R}^2$ by the equivalence relation generated by
- $(p, \theta) \sim (-p, \theta + \pi)$ for all $p, \theta \in \mathbb{R}$.
- $(0, \theta) \sim (0, \theta')$ for all $\theta, \theta' \in \mathbb{R}$.

    (iii) *Natural measure.* An isometry of $\mathbb{R}^2$ takes lines to lines, and thus induces a map $\mathscr{L} \to \mathscr{L}$. The measure $dp\,d\theta$ is invariant under these maps.
    (iv) If C is a regular curve in $\mathbb{R}^2$, let $N_C(p, \theta)$ denote the number of points of intersection of C with the (p, θ)-line.
     (v) *Crofton's formula:* If C is a regular curve of length ℓ then

$$\iint_{\mathscr{L}} N_C(p, \theta)\, dp\,d\theta = 2\ell$$

    (vi) If $\Omega \subset \mathbb{R}^2$ is an open set bounded by a finite union of regular closed curves, let $L_\Omega(p, \theta)$ denote the total length of the intervals of intersection of Ω and the (p, θ) line.
   (vii) Generalized Crofton formula: If a set Ω as above has area A, then

$$\iint_{\mathscr{L}} L_\Omega(p, \theta)\,dp\,d\theta = \pi A$$

   (b) The isoperimetric inequality
      (i) Let Ω be an open set in $\mathbb{R}^2$ bounded by a closed regular curve C, where Ω has area A and C has length L. Then

$$L^2 \geq 4\pi A.$$

      Furthermore, if $L^2 = 4\pi A$ then C is a circle.
     (ii) Corollary: The circle minimizes perimeter among curves enclosing a fixed area.
    (iii) Corollary: The circle maximizes enclosed area among curves with a fixed length.
    (iv) One proof of the isoperimetric inequality uses Crofton's formula to show that the integral of a certain positive real-valued function on $\mathscr{L} \times \mathscr{L}$ is equal to $L^2 - 4\pi A$.
     (v) The difference $L^2 - 4\pi A$ is the *isoperimetric deficit.*

(3) **Surfaces**

   (a) Definition of a regular surface: A subset $S \in \mathbb{R}^3$ such that for each $p \in S$ there is a neighborhood V in $\mathbb{R}^3$ and a map $X : U \to V \cap S$, where $U \subset \mathbb{R}^2$ is open, satisfying:
      (i) X is differentiable
     (ii) X is a homeomorphism
    (iii) X is an immersion, i.e. for each $q \in U$, the differential dX_q is injective.
   (b) Equivalent definitions: Locally, a regular surface is
      (i) The graph of a differentiable function over one of the coordinate planes xy, xz, or yz.
     (ii) The graph of a differentiable function over *some* plane in $\mathbb{R}^3$.
    (iii) The inverse image of a regular value of a differentiable function $F(x, y, z)$.
    (iv) The image of the xy plane under a diffeomorphism from an open set in $\mathbb{R}^3$ to $\mathbb{R}^3$.

(c) Parameterizations: A differentiable map $X : U \to \mathbb{R}^3$ with injective differential at every point is a *immersion* or a *regular parameterized surface*; after restricting to a sufficiently small open set $V \subset U$, the image is a regular surface. In other words, the image of an immersion is locally regular.

(d) The *inverse function theorem*: If $\phi : U \to V$ is a differentiable map and if $d\phi_p$ is an isomorphism ($\Leftrightarrow$ the matrix of partial derivatives at p is invertible), then ϕ is a diffeomorphism *near p*, i.e. there exists a neighborhood U' of p such that $\phi : U' \to V' = \phi(U')$ is a diffeomorphism.

(e) General philosophy: Many ideas from multivariable calculus can be generalized to regular surfaces. Often the generalization is defined like this: Use local coordinates to move everything into $\mathbb{R}^2$, then apply the usual definition for functions of two variables.

(f) *Differentiable functions on surfaces.*
  (i) A function $f : S \to \mathbb{R}$ on a regular surface can be locally expressed as $f(u, v)$, where (u, v) are local coordinates on S near a point $p = (u_0, v_0)$.
  (ii) If $f(u, v)$ is differentiable (in the multivariable calculus sense) at (u_0, v_0), then we say f is *differentiable at p.*
  (iii) This definition does not depend on the coordinate system, since a change of coordinates is differentiable.
  (iv) If f is differentiable at every point of S, then it is *differentiable.*

(g) *Differentiable maps between surfaces.*
  (i) A continuous map $\phi : S_1 \to S_2$ between two regular surfaces can be locally expressed as $f(u, v) = (s(u, v), t(u, v))$, where (u, v) are local coordinates on S_1 near $p = (u_0, v_0)$ and (s, t) are local coordinates on S_2 near $\phi(p)$.
  (ii) If $s(u, v)$ and $t(u, v)$ are differentiable at p, then we say ϕ is *differentiable at p.*
  (iii) If ϕ is differentiable at every point of S_1, then ϕ is *differentiable.*

(h) *Tangent plane.* If $X(u, v)$ is a local parameterization of S, then the span of X_u and X_v at a point p is the *tangent plane of S at p*, denoted T_pS.

(i) An alternate definition of the tangent plane: Consider the set of all curves in S that pass through p. The set consisting of their tangent vectors at p is T_pS.

(j) *Differential.* A differentiable map $\phi : S_1 \to S_2$ induces a linear map $d\phi_p : T_pS_1 \to T_{\phi(p)}S_2$, the *differential of ϕ at p.* In local coordinates (u, v) near p and (s, t) near $\phi(p)$, we can write $\phi(u, v) = (s(u, v), t(u, v))$. Then the differential has matrix

$$d\phi_p = \begin{pmatrix} \frac{\partial s}{\partial u}(p) & \frac{\partial s}{\partial v}(p) \\ \frac{\partial t}{\partial u}(p) & \frac{\partial t}{\partial v}(p) \end{pmatrix}.$$

(k) The *inverse function theorem for surfaces.* If $\phi : S_1 \to S_2$ is a differentiable map and $d\phi_p$ is an isomorphism, then ϕ is a diffeomorphism *near p*, i.e. there exists a neighborhood U' of p such that $\phi : U' \to V' = \phi(U')$ is a diffeomorphism.

(l) A map $\phi : S_1 \to S_2$ whose differential is an isomorphism at every point need not be injective or surjective.
  Examples:
  • The inclusion of a small disk by a coordinate chart (not surjective).
  • The plane mapping to the torus by a doubly-periodic parameterization function (not injective).

(m) Some examples of regular surfaces:
  • A graph $z = f(x, y)$.
  • Inverse image of a regular value $\{(x, y, z) \mid F(x, y, z) = c\}$.
  • *Surface of revolution.* Rotate a plane curve $\beta(t)$ around a line, use t and rotation angle θ as parameters.
  • The surface of revolution of a circle that does not intersect the axis is a *circular torus.*

- A surface that contains a line segment through each of its point is *ruled*. Such a surface can be parameterized by $X(s,t) = \alpha(s) + t\beta(s)$ where α is a space curve and β is a nonzero vector-valued function.
- *Surface of tangents.* $X(s,t) = \alpha(s) + t\alpha'(s)$ where α is a space curve parameterized by arc length. This surface is ruled.
- *Tubes.* Let $X(s,\theta) = \alpha(s) + \epsilon\cos(\theta)n(s) + \epsilon\sin(\theta)b(s)$, where α is a space curve with unit normal n and unit tangent t, and $\epsilon > 0$ is the tube radius.
- The *cone* on the space curve $\alpha(t)$ is parameterized by $X(s,t) = t\alpha(s)$.

(4) **Geometry of surfaces: first and second fundamental forms**

(a) Restricting the inner product of $\mathbb{R}^3$ makes T_pS into an *inner product space*. The associated quadratic form is the *first fundamental form*, denoted I_p. Thus $I_p(w)$ is the squared length of w (as a vector in $\mathbb{R}^3$).

(b) In the basis X_u, X_v for T_pS given by a local parameterization, the matrix of I_p is
$\begin{pmatrix} E & F \\ F & G \end{pmatrix}$ where $E = \langle X_u, X_u \rangle$ $F = \langle X_u, X_v \rangle$ $G = \langle X_v, X_v \rangle$.
In other words, we have "$I = \langle dX, dX \rangle$".

(c) The length of a curve $\alpha(t) = (u(t), v(t))$ on S is given by
$$\int_a^b \sqrt{Eu'^2 + 2Fu'v' + Gv'^2}\,dt.$$

Note that Eu'^2 means $E(u(t), v(t))\,(u'(t))^2$, and similarly for the other terms.

(d) The area of a region Ω contained in a local coordinate chart (u,v) is given by
$$\iint_\Omega \sqrt{EG - F^2}\,du\,dv.$$

Note that when S is contained in $\mathbb{R}^2$, this is the usual formula for change of variables, and $\sqrt{EG - F^2}$ is the Jacobian of the transformation.

(e) The angle θ between vectors $w_1 = aX_u + bX_v$ and $w_2 = cX_u + dX_v$ satisfies
$$\cos(\theta) = \frac{\langle w_1, w_2 \rangle}{|w_1|\,|w_2|} = \frac{Eac + F(ad + bc) + Gbd}{\sqrt{(Ea^2 + 2Fab + Gb^2)(Ec^2 + 2Fcd + Gd^2)}}$$

(f) A map between surfaces whose differential preserves length of vectors is a (local) *isometry*. If the differential preserves angles, then the map is *conformal*. Note than an isometry is conformal.

(g) An *orientation* of a surface is a choice of a unit normal vector at each point in such a way that the resulting map $N : S \to S^2$ is continuous. Here $S^2 = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\}$. If a surface S has an orientation, then it has exactly two, and we say S is *orientable*.

(h) When parameterizing an oriented surface, we always choose $X(u, v)$ so that $X_u \wedge X_v$ is a positive multiple of the unit normal, i.e.
$$N(u, v) = \frac{X_u \wedge X_v}{|X_u \wedge X_v|}$$

(i) The map N is called the *Gauss map* of the surface. The differential of the Gauss map is self-adjoint with respect to I_p.

(j) The *second fundamental form* is the quadratic form II_p on T_pS defined by $II_p(w) = -\langle dN_p(w), w \rangle = \langle \frac{\partial^2 X}{\partial w^2}, N(p) \rangle$. So II_p is the normal component of the acceleration of a path in S with tangent vector w. One could summarize this definition as "$II = -\langle dX, dN \rangle = \langle d^2X, N \rangle$".

(k) The eigenvalues of $-dN_p$ are the *principal curvatures* of S at p, denoted k_1, k_2. The associated eigenspaces are the *principal directions*.

(l) The product of the principal curvatures is the *Gaussian curvature* $K(p) = k_1(p)k_2(p) = \det(dN_p)$.

(m) The average of the principal curvatures is the *mean curvature* $H(p) = \frac{1}{2}(k_1(p) + k_2(p)) = \text{tr}(dN_p)$.

(n) In local coordinates, the matrix of II_p is given by $\begin{pmatrix} e & f \\ f & g \end{pmatrix}$ where:

$$e = \langle X_{uu}, N \rangle = -\langle X_u, N_u \rangle$$
$$f = \langle X_{uv}, N \rangle = -\langle X_u, N_v \rangle = -\langle X_v, N_u \rangle$$
$$g = \langle X_{vv}, N \rangle = -\langle X_v, N_v \rangle$$

(o) This is different from the matrix of dN_p, unless X_u and X_v are orthonormal. In general, we have $dN_p = -\begin{pmatrix} E & F \\ F & G \end{pmatrix}^{-1} \begin{pmatrix} e & f \\ f & g \end{pmatrix}$.

(p) Using the formula for N in terms of X_u and X_v gives the convenient formula

$$e = \frac{1}{\sqrt{EG - F^2}} \det(X_u \ X_v \ X_{uu})$$

and similarly for f and g, replacing only the second derivative term with X_{uv} or X_{vv}, respectively.

(q) Using the formula for dN_p, we have

$$K = \frac{eg - f^2}{EG - F^2} \quad \text{and} \quad H = \frac{Eg - 2Ff + Gg}{2(EG - F^2)}$$

and the principal curvatures are the roots of the polynomial $\lambda^2 - 2H\lambda + K$.

(r) If $\alpha(s)$ is a curve contained in S, then the length of the projection of $\alpha''(s)$ onto $N(\alpha(s))$ is the *normal curvature* of α, denoted k_N. The normal curvature at $\alpha(s)$ only depends on $\alpha'(s)$, and is given by $II_{\alpha(s)}(\alpha'(s))$. Here we assume $\alpha(s)$ is parameterized by arc length.

(s) The principal curvatures at p are the extreme values of the normal curvature as α' varies over all unit tangent vectors at p.

(t) Classification of points on a surface:
- If $K(p) > 0$, then p is an *elliptic point*.
- If $K(p) = 0$ but dN_p is nonzero, then p is a *parabolic point*.
- If $K(p) = 0$ and dN_p is zero, then p is a *planar point*.
- If $K(p) < 0$, then p is a *hyperbolic point*.
- If $k_1(p) = k_2(p)$ (or equivalently, $H(p)^2 = K(p)$), then p is an *umbilic point*.

(u) Typical examples:
- Every point on the unit sphere is elliptic and umbilic.
- Every point on a cylinder is parabolic.
- Every point on a plane is planar
- The point $(0,0,0)$ on $\{z = (x^2 + y^2)^2\}$ is planar.
- The point $(0,0,0)$ on the "saddle" $\{z^2 = x^2 - y^2\}$ is hyperbolic.
- If $f''(x) > 0$, then every point on the surface of rotation of f is hyperbolic.
- The point $(0,0,0)$ on the circular paraboloid $\{z = x^2 + y^2\}$ is umbilic.

(v) A curve in S whose tangent vector at each point is a principal direction is a *line of curvature*.

(w) Special cases:
   (i) If $F = 0$, then the horizontal and vertical lines in the uv plane correspond to orthogonal curves in S.

(ii) If $F = f = 0$, then the principal curvatures are e/E and g/G, the principal directions are X_u and X_v, and the horizontal and vertical lines in the uv plane correspond to lines of curvature in S.

(x) A surface is *minimal* if its mean curvature is identically zero. Examples include:
  - The helicoid $X(s,t) = (0,0,s) + t(\cos(s), \sin(s), 0)$
  - The catenoid, i.e. the surface of revolution of $y = \cosh(x)$ about the x axis

(y) Minimal surfaces are critical points of the area functional, i.e. the derivative of area with respect to any compactly supported variation vector field is zero.

(5) Vector fields, flows, and good parameterizations

(a) A *vector field* on an open set $U \subset \mathbb{R}^n$ is a differentiable map $W : U \to \mathbb{R}^n$; we think of $W(p)$ as a vector based at the point p.

(b) A *flow line* or *integral curve* of a vector field is a curve $\alpha : I \to \mathbb{R}^n$ where $\alpha'(t) = W(\alpha(t))$, i.e. the value of W gives the tangent vector to the path.

(c) Existence of flow lines: Let W be a vector field on U. For any $p \in U$ there exists $\epsilon > 0$ and a flow line $\alpha : (-\epsilon, \epsilon) \to U$ such that $\alpha(0) = 0$.

(d) The local flow: Let W be a vector field on U and let $K \subset U$ be a compact set. Then there exists $\epsilon > 0$ and a map $\Phi : (-\epsilon, \epsilon) \times K \to U$ such that:
  (i) $\Phi(0,p) = p$ for all $p \in K$.
  (ii) $t \mapsto \Phi(t,p)$ is a flow line for any fixed $p \in K$
  Furthermore, this map is unique up to restricting to a smaller value of ϵ.

(e) Version for regular surfaces: A vector field on $S \subset \mathbb{R}^3$ is a differentiable map $W : S \to \mathbb{R}^3$ such that $W(p) \in T_p S$ for all $p \in S$. In local coordinates $X(u,v)$ we can write $W(u,v) = a(u,v)X_u + b(u,v)X_v$, locally identifying W with the vector field $(a(u,v), b(u,v))$ on $\mathbb{R}^2$.

(f) Flow lines, local flow, existence results extend to vector fields on surfaces.

(g) *Good parameterizations.* Using results on vector fields, we obtain the following.
  (i) For any $p \in S$ there is a parameterization $X(u,v)$ of a neighborhood of p that is *orthogonal*, i.e. For all (u,v), we have $\langle X_u, X_v \rangle = 0$. Equivalently, the first fundamental form is diagonal.
  (ii) For any non-umbilic point $p \in S$, there is a parameterization of a neighborhood of p such that the coordinate curves are lines of curvature of S.

(h) *Isothermal coordinates.* For any $p \in S$ there is a local coordinate system $X(u,v)$ that is orthogonal and in which $|X_u| = |X_v| = \lambda(u,v)$. Equivalently, the first fundamental form is $E = G = \lambda^2$, $F = 0$.

(i) The Gaussian curvature in an isothermal coordinate system is given by $K = -(1/\lambda^2)\Delta \log(\lambda)$.

(6) Intrinsic geometry of surfaces

(a) The *Gauss frame* of an oriented surface with local parameterization $X(u,v)$ is the frame $(X_u, X_v, N(u,v))$ where $N(u,v) = (X_u \wedge X_v)/|X_u \wedge X_v|$.

(b) The derivative of the Gauss frame can be expressed in terms of the Gauss frame, giving

$$X_{uu} = \Gamma_{11}^1 X_u + \Gamma_{11}^2 X_v + eN$$
$$X_{uv} = \Gamma_{12}^1 X_u + \Gamma_{12}^2 X_v + fN$$
$$X_{vu} = \Gamma_{21}^1 X_u + \Gamma_{21}^2 X_v + fN$$
$$X_{vv} = \Gamma_{22}^1 X_u + \Gamma_{22}^2 X_v + gN$$

where the coefficients Γ^i_{jk} are the *Christoffel symbols*.

(c) In index notation, let $L_{ij} = \begin{pmatrix} e & f \\ f & g \end{pmatrix}$ and write X_i for $\partial X/\partial u_i$ and X_{ij} for $\partial^2 X/\partial u_i \partial u_j$.
Then
$$X_{ij} = \Gamma^i_{jk} X_i + L_{ij} N.$$

(d) Relations like $\langle X_{uu}, X_u \rangle = \frac{1}{2} E_u$ connect the Christoffel symbols to the first funda-mental form (see (2) on p232). This leads to the formula
$$\Gamma^i_{jk} = \frac{1}{2} g^{i\ell} \left(\frac{\partial g_{j\ell}}{\partial u_k} + \frac{\partial g_{\ell k}}{\partial u_j} - \frac{\partial g_{jk}}{\partial u_\ell} \right)$$
where $g_{ij} = \begin{pmatrix} E & F \\ F & G \end{pmatrix}$ is the first fundamental form and g^{ij} is its inverse matrix.

(e) Equality of mixed third partial derivatives (e.g. $X_{uuv} = X_{uvu}$) gives a system of nine equations relating Christoffel symbols, the first and second fundamental forms, and their derivatives. These equations are of two types:
   (i) Gauss equations: Express the Gaussian curvature K as a function of Christoffel symbols and their first derivatives.
   (ii) Codazzi-Mainardi equations: Express derivatives of second fundamental form (e.g. $e_v - f_u$) in terms of the second fundamental form and the Christoffel symbols (e.q. $e\Gamma^1_{12} + f(\Gamma^2_{12} - \Gamma^1 11) - g\Gamma^2_{11}$).

(f) Gauss equation implies the *Theorema Egregium*: Locally isometric surfaces have the same Gaussian curvature.

(g) Bonnet's fundamental theorem: If E, F, G, e, f, g are differentiable functions and if E, F, G describe a positive definite form at each point, then these arise locally as the first and second fundamental forms of a parameterized surface if and only if they obey the Gauss and Codazzi-Mainardi equations. Furthermore, if two surfaces have the same first and second fundamental form with respect to some local coordinate system, then the surfaces are related by an isometry of $\mathbb{R}^3$.

(h) *The covariant derivative.*
   (i) Let W be a vector field on a regular surface S with local parameterization $X(u, v)$. Then the partial derivatives $\partial W/\partial u$ and $\partial W/\partial v$ may not be tangent to S. The projections of these vectors to the tangent plane are the *covariant derivatives* $DW/\partial u$ and $DW/\partial v$.
   (ii) More generally, if α is a curve in S, then the projection of $\frac{d}{dt}(W(\alpha(t))$ to $T_{\alpha(t)}S$ is the covariant derivative of W along α, denoted DW/dt.
   (iii) The covariant derivative is linear and only depends on the tangent vector to α: If $\alpha'(t) = a(t)X_u + b(t)X_v$ then
   $$\frac{DW}{dt} = a \frac{DW}{\partial u} + b \frac{DW}{\partial v}.$$
   (iv) The covariant derivative DW/dt only depends on the values of a vector field on the curve itself, and is therefore defined for a *vector field along a curve.*
   (v) The tangent vectors $\alpha'(t)$ form a vector field along a curve α. Its covariant derivative is the *acceleration* $D\alpha'(t)/dt$.

(i) *Parallelism.*
   (i) A vector field W along a curve is *parallel* if $DW/dt = 0$.
   (ii) The angle between a pair of parallel vector fields is constant. More generally,
   $$\frac{d}{dt} \langle W, V \rangle = \langle \frac{DW}{dt}, V \rangle + \langle W, \frac{DV}{dt} \rangle.$$
   (iii) Given a vector $W(0) \in T_{\alpha(0)}S$, there is a unique extension to a parallel vector field $W(t)$ along α.

(iv) With $W(0)$ extended to $W(t)$ as above, the value $W(t) \in T_{\alpha(t)}S$ is called the *parallel transport* of $W(0)$ from $\alpha(0)$ to $\alpha(t)$ along α.

(v) If two surfaces are tangent along a curve, then the parallel transport of a vector along the curve can be computed in either surface and the results will be equal.

(j) *Geodesics.*

    (i) A *geodesic* is a curve whose unit tangent vector field is parallel.

    (ii) If $\alpha(s)$ is parameterized by arc length, then $D\alpha'/ds$ is orthogonal to $\alpha'(s)$. Therefore, there is a real number $k_g(s)$ such that

$$\alpha'(s) \wedge \frac{D\alpha'}{ds} = k_g(s)N(\alpha(s))$$

where N is the unit normal vector to the surface. The quantity $k_g(s)$ is the *geodesic curvature* of α. (Its sign depends on the orientation induced by the local parameterization.)

    (iii) Alternate definition of a geodesic: a curve with $k_g \equiv 0$.

    (iv) Given a point $p \in S$ and a unit vector $v \in T_pS$ there is a unique geodesic $\gamma : (-\epsilon, \epsilon) \to S$ such that $\gamma(0) = p$ and $\gamma'(0) = v$.

    (v) In local coordinates (u_1, u_2), the geodesic equations are

$$\frac{d^2 u_i}{dt^2} + \Gamma^i_{jk}\frac{du_j}{dt}\frac{du_k}{dt} = 0$$

    (vi) Geodesic examples:
- Straight lines in $\mathbb{R}^2$
- Great circles on S^2
- Helices on a right circular cylinder

    (vii) On a surface of revolution, let $\theta(s)$ denote the angle between $\alpha'(s)$ and the parallel curve it intersects at $\alpha(s)$, and let $r(s)$ denote the distance from $\alpha(s)$ to the axis of revolution. Then the quantity $r(s)\cos(\theta(s))$ is constant along any geodesic. This is *Clairaut's relation.*

    (viii) It follows that a meridian on a surface of revolution (i.e. curve of intersection of the surface with a plane containing its axis) is a geodesic.

    (ix) A parallel on a surface of revolution is a geodesic if and only if it is a local maximum of distance to the axis.

(7) **The Gauss-Bonnet Theorem**

(a) *Regions and curves.*

    (i) A *piecewise smooth curve* on a surface has a pair of tangent vectors at each of its vertices. The angle between these is an *exterior angle* of the curve.

    (ii) A *simple region* on a surface is a closed set homeomorphic to a disk bounded by a piecewise smooth simple closed curve.

    (iii) A *regular region* on a surface is a compact set whose boundary is a finite disjoint union of piecewise smooth simple closed curves.

(b) *Gauss-Bonnet v1.0.* If $R \subset S$ is a simple region with smooth boundary that is contained in a single coordinate chart of the surface, then

$$\int_{\partial R} k_g(s)ds + \iint_R KdA = 2\pi.$$

(c) For an orthogonal coordinate system, the proof of the Gauss-Bonnet theorem has two ingredients:

    (i) The 2-dimensional divergence theorem

(ii) The *theorem of the turning tangents*: If $\gamma : [0, \ell] \to \mathbb{R}^2$ is a smooth simple closed curve parameterized by arc length with signed curvature κ, then

$$\int_0^\ell \kappa(s)ds = \pm 2\pi$$

where the sign indicates whether the disk bounded by γ lies to the left $(+)$ or to the right $(-)$ as γ is traversed in the direction of increasing s.

(d) *Gauss-Bonnet v2.0.* If $R \subset S$ is a simple region with piecewise smooth boundary that is contained in a single coordinate chart of the surface, and if $\theta_1, \ldots, \theta_n$ are the external angles of the positively oriented boundary of R, then

$$\sum_{i=1}^n \theta_i + \int_{\partial R} k_g(s)ds + \iint_R KdA = 2\pi.$$

(e) A *triangulation* of a regular region is a decomposition into triangles (simple regions with three vertices) such that the triangles are pairwise disjoint unless they share exactly one vertex or exactly one edge.

(f) If R is a triangulated regular region with V vertices, E edges, and F triangles, then the Euler characteristic of R is the quantity

$$\chi(R) = V - E + F.$$

The Euler characteristic depends only on R and not on the particular triangulation.

(g) Any regular region has a triangulation such that each triangle is contained in a single coordinate chart.

(h) *Gauss-Bonnet v3.0.* If R is a regular region on a surface S (e.g. $R = S$ if S is compact) whose positively oriented boundary has external angles $\theta_1, \ldots, \theta_n$, then we have

$$\sum_{i=1}^n \theta_i + \int_{\partial R} k_g(s)ds + \iint_R KdA = 2\pi\chi(R).$$

(i) *The classification of compact orientable surfaces.*

   (i) The *genus* of a compact orientable surface is a nonnegative integer that uniquely determines its homeomorphism type, i.e. two surfaces are homeomorphic if and only if they have the same genus.

   (ii) The sphere has genus $g = 0$, the torus has $g = 1$.

   (iii) Informally, the genus $g(S)$ is the number of "handles" that one must attach to the sphere in order to create a surface homeomorphic to S.

   (iv) Let $C = \cup_{i=1}^g C_i$ where C_i is the circle in $\mathbb{R}^2$ with center $(2i - 1, 0)$ and radius 1. Let S be the set of points in $\mathbb{R}^3$ whose distance from C is exactly $1/4$. Then S is a compact (topological) surface of genus g.

(j) Corollaries of GB v3.0.

   (i) If S is a compact orientable regular surface, then

$$\iint_S KdA = 2\pi\chi(S).$$

   (ii) Gauss-Bonnet v2.1: The restriction that R must lie in a single coordinate chart can be dropped from v2.0.

   (iii) If a regular surface has positive Gaussian curvature, then it is homeomorphic to the sphere.

   (iv) If $S \subset \mathbb{R}^3$ is a compact regular surface that is not homeomorphic to the sphere, then S contains points of positive, negative, and zero Gaussian curvature.

   (v) On a surface of nonpositive Gaussian curvature, a pair of geodesic segments joining points p and q never bound a disk. (Compare: Any two geodesic segments joining the north and south poles of S^2 bound a topological disk.)

(8) **Exponential map and normal coordinates**

(a) Fix a regular surface $S \subset \mathbb{R}^3$ and let TS denote the set of pairs (p, v) where $p \in S$ and $v \in T_pS$. This is the *tangent bundle* of S.

(b) A local coordinate system $X(u, v)$ for S gives a local coordinate system (u, v, ξ, ζ) for TS, where

$$(u, v, \xi, \zeta) \mapsto (X(u, v), \xi X_u + \zeta X_v).$$

(c) The geodesic equations define a vector field on TS with local flow $\Phi : (-\epsilon, \epsilon) \times TK \to TS$ where $TK = \{(p, v) \in TS \mid p \in K\}$ and $K \subset S$ is compact. This is the *geodesic flow*.

(d) The geodesic flow has the homogeneity property $\Phi(ct, p, v) = \Phi(t, p, cv)$ for any $c \in \mathbb{R}$. Thus for small v the flow is defined for a large time interval.

(e) Let $\pi : TS \to S$ be the map defined by $\pi(p, v) = p$. The *exponential map* at $p \in S$ is defined by

$$\exp_p(v) = \pi \circ \Phi(1, p, v)$$

i.e. send $v \in T_pS$ to the endpoint of the geodesic segment of length $|v|$ starting at p in the direction of v.

(f) The map $\exp_p$ is defined on an open subset of T_pS that contains $(0, 0)$. Thus there exists δ such that $\exp_p$ is defined for all $v \in T_pS$ with $|v| < \delta$.

(g) The differential of $\exp_p$ at 0 is the identity, i.e.

$$(d \exp_p)_0(v) = v$$

for any $v \in T_pS \simeq T_0 T_pS$.

(h) By the IFT, the exponential map is a local diffeomorphism. The diffeomorphic image of a neighborhood of $0 \in T_pS$ is a *normal neighborhood* of p.

(i) Composing orthonormal rectangular coordinates for T_pS and $\exp_p$ gives *normal coordinates* for S centered at p.

(j) Composing polar coordinates for T_pS and $\exp_p$ gives *geodesic polar coordinates* for S centered at p.

(k) In normal coordinates, $E(p) = G(p) = 1$ and $F(p) = 0$.

(l) In geodesic polar coordinates, $E \equiv 1$, $F \equiv 0$, and G satisfies

$$\lim_{\rho \to 0} G(\rho, \theta) = 0 \qquad \lim_{\rho \to 0} \frac{\partial}{\partial \rho} \sqrt{G(\rho, \theta)} = 1.$$

(Compare these to polar coordinates in $\mathbb{R}^2$.)

(m) Theorem: For all $p \in S$ there is a neighborhood U of p that is also a normal neighborhood of q for all $q \in U$.

(n) For all $p \in S$ there is a normal neighborhood U of p such that any geodesic segment in U is *minimizing*, i.e. its length is less than or equal to that of any path in S with the same endpoints.

(o) Theorem: For all $p \in S$ there exists a normal neighborhood $U \in S$ that is *convex*, i.e. for all $q_1, q_2 \in U$ there is a unique geodesic segment with endpoints q_1, q_2 that is contained in U.

(p) Corollary: Every compact surface has a finite geodesic triangulation (i.e. a triangulation with finitely many triangles, each of which is bounded by three geodesic segments).