Math 442 – Differential Geometry of Curves and Surfaces

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This note gives a short solution to the "hard part" of problem 3.5.11 from the textbook. It is also possible to solve the problem by a tedious calculation of E, F, G and e, f, g for the parallel surface.

Let S be a regular oriented surface in space with local parameterization $X(u, v)$ and normal $N(u, v)$. Let H and K denote the mean and Gaussian curvature functions of S.

Theorem 1. *The parallel surface S_a with local parameterization $Y(u, v) = X(u, v) + aN(u, v)$ has Gaussian curvature*

$$\bar{K} = \frac{K}{1 - 2Ha + Ka^2}$$

and mean curvature

$$\bar{H} = \frac{H - Ka}{1 - 2Ha + Ka^2}.$$

Proof of theorem. Since N_u and N_v are tangent to S, it follows that Y_u and Y_v are also tangent to S. Therefore the surfaces S and S_a have parallel tangent spaces and their respective normals at $X(u, v)$ and $Y(u, v)$ are equal. In terms of shape operators, this means

$$d\bar{N}(Y_u) = dN(X_u)$$
$$d\bar{N}(Y_v) = dN(X_v)$$

Let $\bar{A}$ be the matrix of $d\bar{N}$ with respect to the basis (Y_u, Y_v) and let A be the matrix of dN with respect to (X_u, X_v). Then the above equations correspond to the single matrix equation

$$\bar{A} = WA$$

where W is the matrix of change of basis from (X_u, X_v) to (Y_u, Y_v). We calculate

$$Y_u = X_u + aN_u = X_u + adN(X_u)$$
$$Y_v = X_v + aN_v = X_v + adN(X_v)$$

which gives

$$W^{-1} = (I + aA).$$

(To explain the inverse here, note that to change bases from $\{v_i\}$ to $\{w_i\}$, one must write w_i in terms of v_i and put the coefficients in a matrix. Thus the previous equations give the change of basis from (Y_u, Y_v) to (X_u, X_v), and we invert to get W.)

Combining the previous calculations we have $\bar{A} = (I + aA)^{-1}A$. The eigenvalues of A are $\{-k_1, -k_2\}$ and thus $(I + aA)^{-1}$ has eigenvalues $\{(1 - ak_1)^{-1}, (1 - ak_2)^{-1}\}$ and *the same eigenspaces*. Multiplying matrices with the same eigenspaces multiplies eigenvalues, so the eigenvalues of $\bar{A}$ are

$$\frac{-k_1}{1 - ak_1} \quad \text{and} \quad \frac{-k_2}{1 - ak_2}.$$

Since $\bar{K}$ is the product of these eigenvalues and $-\frac{1}{2}\bar{H}$ is their average, the formulas claimed in the statement of the theorem follow by algebra. □