Math 442 - Differential Geometry of Curves and Surfaces

Challenge Problems

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(C1) Describe all curves on the unit sphere with constant torsion. Are any of them closed? (Hint: Begin with the case $\tau = 0$.)

(C2) Generalize the Frenet frame and the Frenet equations to $\mathbb{R}^4$ as follows.

Let $\alpha : I \to \mathbb{R}^3$ be a differentiable curve (not necessarily parameterized by arc length) such that for all $t \in I$, the vectors $(\alpha'(t), \alpha''(t), \alpha'''(t))$ form a basis of $\mathbb{R}^3$.

(a) Apply the Gram-Schmidt algorithm to this basis to obtain an orthonormal basis; show that the Frenet vectors, curvature, and torsion all appear as part of this calculation.

(b) Now suppose $\alpha : I \to \mathbb{R}^4$ is a differentiable curve. Use the results of (a) to generalize the Frenet frame to this case, obtaining an orthonormal basis of $\mathbb{R}^4$ adapted to the curve. Find three curvature-like functions that play the same role as κ, τ in the 3-dimensional case, and write the four Frenet differential equations that the frame vectors obey.

(C3) Let $\alpha : I \to \mathbb{R}^3$ be a differentiable curve parameterized by arc length, with curvature $\kappa_\alpha(s) \neq 0$ and torsion $\tau_\alpha(s)$. For each $s \in I$, let $\beta(s)$ denote the center of the osculating circle of α at $\alpha(s)$.

(a) Compute the speed $|\beta'(s)|$, curvature $\kappa_\beta(s)$, and torsion $\tau_\beta(s)$ of the curve β. (*Warning: s is not necessarily the arc length parameter for β!*)

(b) Find a particular curve α so that the new curve β is "the same", i.e. related to α by a rotation and/or translation.

(C4) (a) Let $\alpha : I \to \mathbb{R}^3$ be a differentiable curve that lies on the unit sphere (i.e. $|\alpha(s)| = 1$ for all $s \in I$). Show that $\kappa(s) \geq 1$ for all $s \in I$.

(b) Suppose instead that α lies on the ellipsoid $ax^2 + by^2 + cz^2 = 1$, where $a, b, c > 0$. What is the minimum possible value for the curvature?

(C5) Let $\alpha : I \to R^2$ be a differentiable plane curve with positive, increasing curvature (i.e. $\kappa(s), \kappa'(s) > 0$).

(a) Show that the osculating circles of α are nested, meaning that if $s' > s$, then the osculating circle at $\alpha(s')$ is contained in the osculating cicle at $\alpha(s)$.

(b) From (a) it follows that the osculating circles fill an open set $U \subset \mathbb{R}^2$ that contains the trace of α. Let V be the unit vector field in U that is tangent to these circles, and which always points counter-clockwise. Then each of the osculating circles is tangent to this V, as is the curve α itself. Why is this surprising? What is going on?

Hint: A curve whose tangent vector is always horizontal is a horizontal line. There is only one horizontal line through any given point.

(C6) For every natural number g, find a polynomial $P_g(x, y, z)$ with the property that $P_g^{-1}(1)$ is a regular surface in $\mathbb{R}^3$ of genus g. (In particular, you should come up with some way to recognize the genus of a closed surface in $\mathbb{R}^3$.)

(C7) What is the minimum degree of a polynomial $P(x, y, z)$ such that $P^{-1}(1)$ is a regular surface of positive genus?

(C8) Let $f(x, y)$ be a smooth function of two variables. How can you use f and its derivatives to determine whether or not $z = f(x, y)$ is (locally) a ruled surface? (Your condition should be *pointwise*, meaning that you are not allowed to compare values or derivatives at different points. Hint: If (x_0, y_0) is a local maximum or local minimum, then the graph cannot be ruled.)

(C9) Let $\alpha(\theta)$ and $\beta(\theta)$ denote a pair of circles in $\mathbb{R}^3$ parameterized with constant speed by $\theta \in [0, 2\pi]$. (Note that a circle in $\mathbb{R}^3$ is defined as the set of all points in a plane that lie a fixed distance from some point in that plane.) Describe the scroll generated by α and β in a way that does not depend on a parameterization, e.g. find a function $F(x, y, z)$ so that the scroll consists of points satsifying $F(x, y, z) = 0$.

(C10) Consider the parabola $P = \{y = x^2\}$ in the plane. Let $p(s)$ denote an arc length parameterization of P with $p(0) = (0, 0)$ and $p'(0) = (1, 0)$. Let $T(s)$ be the tangent line to P at $p(s)$. For any $s \in \mathbb{R}$, apply a rotation and translation so that $p(s)$ is sent to $(s, 0)$ and $T(s)$ becomes the x axis; call the resulting parabola $P(s)$. We say $P(s)$ is the result of *rolling P along the x axis.*

Given a point q in $\mathbb{R}^2$, we can form a path $q(s)$ by applying the same rotation and translation to q as is used to transform P into $P(s)$. (Think of q as being "rigidly attached" to P, so it moves as P rolls.) Show that if $q = (0, 1/4)$, then $q(s)$ is a catenary.

(C11) Find a parameterization of the path traced out by one focus of an elliptical object as it rolls along the x axis without slipping. Your parameterization may need to use functions defined in terms of integrals that cannot be evaluated explicitly.

(C12) Show that the surface of rotation of the curve described in the previous problem has constant mean curvature.

(C13) Show that any embedded curve in $\mathbb{R}^2$ with closed image is a flow line of a vector field. That is, let $\alpha : [0, 1] \to \mathbb{R}^2$ be an injective differentiable map with $\alpha'(t) \neq 0$ for $t \in [0, 1]$. Show that there is a vector field W defined on a neighborhood of $\alpha([0, 1])$ such that $\alpha((0, 1))$ is a flow line of W. (*The **local** version of this problem is P9 on the weekly problem list.*)

(C14) Let $S \subset \mathbb{R}^3$ be a regular surface with no umbilic points, and let $\alpha : I \to S$ be a line of curvature of S corresponding to the principal curvature function k_1. Consider α as a space curve and calculate its curvature in terms of k_1, k_2, and their covariant derivatives.