05200301

Math 520 Final Exam

Spring 2008

Sections 1–3 (Xiao/Dumas/Liaw)

Read these instructions carefully.

- Write your name, section, and "Math 520 Final Exam" on the front of an exam book.

10am / Xiao	=	Section 1
Noon / Dumas	=	Section 2
1pm / Liaw	=	Section 3

- Read each problem carefully before you attempt to solve it.
- Write your solutions to the problems in the examination book.
- Clearly indicate where your solution to each problem begins and ends.
- Make sure your solutions are clear, concise, and legible.
- There are **9** problems on the exam, worth a total of 50 points. the number of points assigned to each problem or part thereof is listed in the right margin.
- **Manage your time carefully. If you get stuck on one problem, move on to another.**

Do not turn the page until you are told to do so!

1. Let P denote the projection onto the column space $C(H)$, where [3 pts]

$$H = \begin{pmatrix} 200820082008 & 520520520520 & 888923164781 \\ 789847589127441 & \sqrt{941} & 389734581 \\ 31897581128 & 17895789243 & 524089484923 \\ 0 & 0 & 555 \\ 222 & 0 & 0 \\ 0 & 111 & 0 \end{pmatrix}.$$

Compute $\mathrm{tr}(P)$ and $\det(P)$, and explain your reasoning.

2. Let F denote the projection from $\mathbb{R}^4$ onto the subspace of vectors satisfying $x_1+x_2+x_3+x_4 = 0$.
   (a) Compute the matrix F explicitly. (That is, write the 4×4 matrix F as your answer, and justify your computation.) [2 pts]
   (b) Compute reduced row echelon form of F. [2 pts]
   (c) What is the rank of F? [1 pt]
   (d) What vector in $C(F)$ is closest to $\boldsymbol{b} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$? [2 pts]

   In other words, find $\boldsymbol{p} \in C(F)$ for which $\|\boldsymbol{b} - \boldsymbol{p}\|$ is as small as possible.

3. Let P_4 denote the vector space of polynomials in one variable of degree less than or equal to 4.
   (a) Let S be the subset of P_4 consisting of all polynomials $p(x)$ that take on the same values at $x = 0$ and $x = 1$, i.e. [2 pts]

   $$S = \{\, p(x) \in P_4 \mid p(0) = p(1) \,\}.$$

   Is S a subspace of P_4? Either show that it is, or explain why it is not.
   (b) Do these four vectors (below) span the space P_4? *Justify your answer.* [2 pts]

   $$\begin{aligned} p_1(x) &= x^4 + x^3 + 2x^2 + 3x + 1 \\ p_2(x) &= x^4 + 2x^3 + 3x^2 + x + 1 \\ p_3(x) &= 2x^4 + 3x^3 + x^2 + x + 1 \\ p_4(x) &= 4x^4 + 6x^3 + 6x^2 + 5x + 3 \end{aligned}$$

   (c) Are the vectors p_1, p_2, p_3, p_4 from part (b) linearly independent? *Justify your answer.* [2 pts]

$* * *$ **The exam continues on the next page.** $* * *$

4. Let $A = \begin{pmatrix} -2 & -2 & 8 \\ -2 & 7 & -10 \\ 8 & -10 & 4 \end{pmatrix}$.

(a) Show that the determinant of A is equal to zero. [2 pts]

(b) Show that vector $\begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix}$ is an eigenvector of A. [2 pts]

(c) Use the trace formula to find the third eigenvalue of A. [1 pt]

(d) Find an orthogonal matrix Q that diagonalizes A. (Write $A = Q\Lambda Q^T$.) [2 pts]

5. Let $K = \begin{pmatrix} 2 & -2 & -1 \\ 1 & -1 & -1 \\ t & -2 & 0 \end{pmatrix}$ depend on the real parameter t.

(a) For which values of t is K invertible? [2 pts]

(b) Show that 1 is an eigenvalue of K for all real t. [2 pts]

(c) Show that K is not diagonalizable for $t = 2$. [2 pts]

6. Determine whether or not each of these matrices is positive definite. Justify your answer in each case.

(a) $\begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 2 \\ 3 & 2 & 1 \end{pmatrix}$ [2 pts]

(b) $\begin{pmatrix} 3 & 2 & 1 \\ 2 & 3 & 2 \\ 1 & 2 & 3 \end{pmatrix}$ [2 pts]

(c) $\begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 10 & 1 & 0 \\ 0 & 1 & 100 & 1 \\ 0 & 0 & 1 & 1000 \end{pmatrix}$ [2 pts]

$* * *$ **The exam continues on the next page.** $* * *$

7. This question concerns the matrix

$$B = \begin{pmatrix} 3 & 0 & 0 & 1 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 2 & 0 \\ 1 & 0 & 0 & 3 \end{pmatrix}.$$

(a) Find all of the eigenvalues of B. [2 pts]

(b) For each eigenvalue, find a maximal set of linearly independent eigenvectors. (Clearly indicate which eigenvectors are associated with each eigenvalue.) [2 pts]

(c) Is B diagonalizable? Why or why not? [1 pt]

(d) Find the solution $\boldsymbol{u}(t)$ to the following differential equation and initial condition: [2 pts]

$$\frac{d\boldsymbol{u}}{dt} = B\boldsymbol{u} \quad \text{and} \quad \boldsymbol{u}(0) = \begin{pmatrix} 1 \\ 1 \\ 0 \\ -1 \end{pmatrix}$$

8. Suppose T is a linear transformation from V to W, and that its matrix with respect to the input basis $\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3$ of V and output basis $\boldsymbol{w}_1, \boldsymbol{w}_2$ of W is

$$J = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 1 \end{pmatrix}.$$

(a) Is the vector $\boldsymbol{v}_1 + \boldsymbol{v}_2 - \boldsymbol{v}_3$ in the kernel of T? [2 pts]

(b) What is the range of T? [2 pts]

(c) Find the matrix J' which represents T with respect to the input basis [2 pts]

$$\begin{aligned} \boldsymbol{v}_1' &= \boldsymbol{v}_1 + \boldsymbol{v}_2 \\ \boldsymbol{v}_2' &= \boldsymbol{v}_2 + \boldsymbol{v}_3 \\ \boldsymbol{v}_3' &= \boldsymbol{v}_1 + \boldsymbol{v}_3 \end{aligned}$$

and output basis

$$\begin{aligned} \boldsymbol{w}_1' &= \boldsymbol{w}_2 \\ \boldsymbol{w}_2' &= \boldsymbol{w}_1 + \boldsymbol{w}_2 \end{aligned}$$

9. Give an example of a 5×5 matrix that is orthogonal, symmetric, positive definite, invertible, and diagonalizable. (Write *one* matrix that has all of these properties.) [2 pts]

Hint: Look for an easy example; do not spend a lot of time or effort trying to construct the matrix!

$***$ **This is the end of the exam.** $***$