Note: Some of these solutions include more explanation than was necessary for full credit.

1. (5 points) This question concerns the matrix

$$B = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 6 \end{pmatrix}.$$

Solution: For use in solving (a)-(c) below, put B^T into row reduced echelon form:

$$R' = \text{RREF}(B^T) = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Note: An alternate solution to this problem might use the RREF of B and the elimination matrix E.

(a) (2 points) Find a basis for $C(B^T)$, the row space of B.

Solution: The pivot columns of B^T form a basis of $C(B^T)$:

$$\begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}, \begin{pmatrix} 2 \\ 3 \\ 4 \\ 5 \end{pmatrix}.$$

In fact, any two columns of B^T (i.e. rows of B) form a basis of $C(B^T)$. (For some other rank 2 matrices, one would need to choose the pair more carefully.)

(b) (2 points) Find a basis for $N(B^T)$, the left null space of B.

Solution: The null space of B^T is one-dimensional, and a basis is computed from the free column of R':

$$\begin{pmatrix} -F \\ I \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$$

To check this, notice that the sum of the first and last rows of B is equal to twice the middle row.

(c) (1 point) Of the four fundamental subspaces associated to B, which one can also be described as $C(B^T)^\perp$, the orthogonal complement of the row space?

Solution: The **null space**, $N(B)$, is the orthogonal complement of $C(B^T)$. The orthogonality of rows of B and vectors $\boldsymbol{x} \in N(B)$ is expressed in the definition of the null space, $B\boldsymbol{x} = \mathbf{0}$.

2. (6 points) Evaluate each determinant, or explain why it is not defined.

(a) (1 point) $\det(-3)$

Solution: -3

(b) (1 point) $\det \begin{pmatrix} 0 & 5 \\ 5 & 27 \end{pmatrix}$

Solution: -25

(c) (1 point) $\begin{vmatrix} 2 & 2 & 0 \\ 1 & 2 & 0 \end{vmatrix}$

Solution: Not defined, because the matrix is rectangular.

(d) (1 point) $\begin{vmatrix} 0 & 1 & 0 & 0 & 7 \\ 1 & 0 & 0 & 0 & 7 \\ 0 & 0 & 1 & 0 & 7 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 2 \end{vmatrix}$

Solution: -1 (One possible shortcut: swap rows 1,2 and subtract row 4 from row 5 to obtain upper triangular with all 1s on the diagonal. Hence $-\det = 1$.)

(e) (1 point) $\det \left[\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right]$

Solution: 1

(f) (1 point) $\det A$, where A is the 4×4 matrix with entries $a_{ij} = \begin{cases} 1, & \text{if } i + j = 5, \\ 0, & \text{otherwise.} \end{cases}$

Solution:

$$\begin{vmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{vmatrix} = 1$$

(Two row exchanges to obtain the identity.)

3. (5 points) The matrix $A = \begin{pmatrix} 1 & 0 & 2 & 1 \\ 2 & 1 & 1 & 2 \\ 4 & 2 & 1 & 1 \\ 6 & 3 & 1 & 2 \end{pmatrix}$ has determinant $|A| = -2$.

(a) (1 point) Is A invertible? *Explain your answer.*

Solution: Yes, because the determinant is nonzero. A matrix is singular if and only if its determinant is zero.

(b) (1 point) Is there any $\boldsymbol{b} \in \mathbb{R}^4$ such that $A\boldsymbol{x} = \boldsymbol{b}$ does not have a solution? If so, give an example. If not, explain why.

Solution: No, there is always a solution, because A is invertible. In fact, $\boldsymbol{x} = A^{-1}\boldsymbol{b}$ is the unique solution.

(c) (3 points) Use Cramer's rule to compute x_4 such that $A \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 3 \end{pmatrix}$.

Solution: Using a column operation and two cofactor expansions:

$$|B_4| = \begin{vmatrix} 1 & 0 & 2 & 1 \\ 2 & 1 & 1 & 0 \\ 4 & 2 & 1 & 0 \\ 6 & 3 & 1 & 3 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 3 & 1 & 3 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 0 \\ 2 & 1 & 0 \\ 3 & 1 & 3 \end{vmatrix} = 3 \begin{vmatrix} 1 & 1 \\ 2 & 1 \end{vmatrix} = -3.$$

Therefore,

$$x_4 = \frac{|B_4|}{|A|} = \frac{-3}{-2} = \frac{3}{2}$$

4. (7 points) This question concerns the matrix $D = \begin{pmatrix} 1 & 1 \\ 1 & 0 \\ 1 & 1 \end{pmatrix}$ and the vector $\boldsymbol{b} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$.

(a) (3 points) Find **three** orthonormal vectors $\boldsymbol{q}_1, \boldsymbol{q}_2, \boldsymbol{q}_3$ in $\mathbb{R}^3$ such that $C(D) = \operatorname{Span}(\boldsymbol{q}_1, \boldsymbol{q}_2)$.

Solution: Apply Gram-Schmidt to the columns of the invertible matrix

$$D' = \begin{pmatrix} \boldsymbol{a}_1 & \boldsymbol{a}_2 & \boldsymbol{a}_3 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{pmatrix}$$

and thus obtain orthonormal vectors $\boldsymbol{q}_1, \boldsymbol{q}_2, \boldsymbol{q}_3$ with the first two spanning $C(D)$.

$$\boldsymbol{A}_1 = \boldsymbol{a}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$\boldsymbol{A}_2 = \boldsymbol{a}_2 - \frac{\boldsymbol{A}_1^T \boldsymbol{a}_2}{\boldsymbol{A}_1^T \boldsymbol{A}_1} \boldsymbol{A}_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} - \frac{2}{3} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \frac{1}{3} \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$$

$$\boldsymbol{A}_3 = \boldsymbol{a}_3 - \frac{\boldsymbol{A}_1^T \boldsymbol{a}_3}{\boldsymbol{A}_1^T \boldsymbol{A}_1} \boldsymbol{A}_1 - \frac{\boldsymbol{A}_2^T \boldsymbol{a}_3}{\boldsymbol{A}_2^T \boldsymbol{A}_2} \boldsymbol{A}_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} - \frac{1}{3} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} - \frac{1}{6} \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

$$\boldsymbol{q}_1 = \frac{1}{\sqrt{3}}\begin{pmatrix}1\\1\\1\end{pmatrix} \quad \boldsymbol{q}_2 = \frac{1}{\sqrt{6}}\begin{pmatrix}1\\-2\\1\end{pmatrix} \quad \boldsymbol{q}_3 = \frac{1}{\sqrt{2}}\begin{pmatrix}-1\\0\\1\end{pmatrix}$$

(b) (2 points) There is no solution to $D\boldsymbol{x} = \boldsymbol{b}$. Find the least squares approximate solution $\hat{\boldsymbol{x}}$.

Solution:

$$\hat{\boldsymbol{x}} = (D^TD)^{-1}D^T\boldsymbol{b} = \left[\begin{pmatrix}1&1&1\\1&0&1\end{pmatrix}\begin{pmatrix}1&1\\1&0\\1&1\end{pmatrix}\right]^{-1}\begin{pmatrix}1&1&1\\1&0&1\end{pmatrix}\begin{pmatrix}1\\2\\3\end{pmatrix} = \begin{pmatrix}3&2\\2&2\end{pmatrix}^{-1}\begin{pmatrix}6\\4\end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix}2&-2\\-2&3\end{pmatrix}\begin{pmatrix}6\\4\end{pmatrix} = \begin{pmatrix}2\\0\end{pmatrix}$$

As a check, notice that the error vector $\begin{pmatrix}1\\2\\3\end{pmatrix} - \begin{pmatrix}2\\2\\2\end{pmatrix} = \begin{pmatrix}-1\\0\\1\end{pmatrix}$ is orthogonal to both columns of D, as it should be.

(c) (1 point) What is the projection of $\boldsymbol{b}$ onto $C(D)$?

Solution:

$$\boldsymbol{p} = D\hat{\boldsymbol{x}} = \begin{pmatrix}2\\2\\2\end{pmatrix}$$

(d) (1 point) What is the projection of $\boldsymbol{b}$ onto $C(D)^{\perp}$?

Solution:

$$\boldsymbol{e} = \boldsymbol{b} - \boldsymbol{p} = \begin{pmatrix}-1\\0\\1\end{pmatrix}$$

5. (6 points)

(a) (1 point) Suppose Q is an orthogonal $n \times n$ matrix. Is $\lambda = 0$ an eigenvalue of Q? *Explain your answer.*

Solution: A matrix with zero as an eigenvalue is singular (since the eigenvector is in the null space), but an orthogonal matrix is invertible (in fact, $Q^{-1} = Q^T$). Therefore, zero is not an eigenvalue of Q.

(b) (3 points) Find the eigenvalues and eigenvectors of the matrix $\begin{pmatrix} 5 & 0 & 1 \\ 0 & 5 & 0 \\ 1 & 0 & 5 \end{pmatrix}$.

Solution: This matrix is $5I + A$, where $A = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$.

Clearly $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ is in the null space of A, so it is an eigenvector of A with $\lambda = 0$.

Since the matrix A switches the first and last components of a vector (while setting the middle equal to zero), it also has eigenvalues $\lambda = 1$ and $\lambda = -1$ with eigenvectors $\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$. These are similar to the case of the 2×2 permutation matrix P_{12}.

Adding $5I$ to a matrix keeps the eigenvectors the same but adds 5 to each eigenvalue. Thus for the matrix in question:

λ	4	5	6
$\boldsymbol{x}$	$\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$	$\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$	$\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$

As a check, notice that the sum of the eigenvalues is 15, and the product is 120, which agree with the trace and determinant of the given matrix (respectively).

Note: One could also find these eigenvalues and eigenvectors by the usual algorithm—computing the characteristic polynomial, finding its roots, then finding bases for the eigenspaces. The solution given here is a possible shortcut, based on knowledge of the 2×2 permutation matrix.

(c) (2 points) Let V be the subspace of $\mathbb{R}^4$ consisting of all vectors orthogonal to $\begin{pmatrix} 15\sqrt{2} \\ -520 \\ 2008 \\ \sqrt{17} \end{pmatrix}$.

Let P be the 4×4 projection matrix onto V. Compute $\operatorname{tr}(P)$ and $\det(P)$, and explain your reasoning.

Solution: Note that $\dim(V) = 3$. Since P is a projection onto a 3-dimensional subspace of $\mathbb{R}^4$, its eigenvalues are $0, 1, 1, 1$ (listed with multiplicity). Therefore

$$\operatorname{tr}(P) = 0 + 1 + 1 + 1 = 3$$
$$\det(P) = 0 \cdot 1 \cdot 1 \cdot 1 = 0.$$

Of course it would be impractical to calculate these (by hand) from the actual matrix

$$P = \frac{1}{4302931} \begin{pmatrix} 4302481 & 7800\sqrt{2} & -30120\sqrt{2} & -15\sqrt{34} \\ 7800\sqrt{2} & 4032531 & 1044160 & 520\sqrt{17} \\ -30120\sqrt{2} & 1044160 & 270867 & -2008\sqrt{17} \\ -15\sqrt{34} & 520\sqrt{17} & -2008\sqrt{17} & 4302914 \end{pmatrix}.$$