Math 520 Exam 2

Spring 2008

Sections 1–3 (Xiao/Dumas/Liaw)

Read these instructions carefully.

- Write your name, section, and "Math 520 Exam 2" on the front of an exam book.

10am / Xiao	=	Section 1
Noon / Dumas	=	Section 2
1pm / Liaw	=	Section 3

- Read each problem carefully before you attempt to solve it.
- Write your solutions to the problems in the examination book.
- Clearly indicate where your solution to each problem begins and ends.
- Make sure your solutions are clear, concise, and legible.
- Each of the **5** problems will contribute approximately the same amount to your grade.
- **Manage your time. If you get stuck on one problem, move on to another.**

Do not turn the page until you are told to do so!

1. This question concerns the matrix

$$B = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 6 \end{pmatrix}.$$

   (a) Find a basis for $C(B^T)$, the row space of B.

   (b) Find a basis for $N(B^T)$, the left null space of B.

   (c) Of the four fundamental subspaces associated to B, which one can also be described as $C(B^T)^\perp$, the orthogonal complement of the row space?

2. Evaluate each determinant, or explain why it is not defined.

   (a) $\det(-3)$

   (b) $\det \begin{pmatrix} 0 & 5 \\ 5 & 27 \end{pmatrix}$

   (c) $\begin{vmatrix} 2 & 2 & 0 \\ 1 & 2 & 0 \end{vmatrix}$

   (d) $\begin{vmatrix} 0 & 1 & 0 & 0 & 7 \\ 1 & 0 & 0 & 0 & 7 \\ 0 & 0 & 1 & 0 & 7 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 2 \end{vmatrix}$

   (e) $\det \left[\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right]$

   (f) $\det A$, where A is the 4×4 matrix with entries $a_{ij} = \begin{cases} 1, & \text{if } i + j = 5, \\ 0, & \text{otherwise.} \end{cases}$

3. The matrix $A = \begin{pmatrix} 1 & 0 & 2 & 1 \\ 2 & 1 & 1 & 2 \\ 4 & 2 & 1 & 1 \\ 6 & 3 & 1 & 2 \end{pmatrix}$ has determinant $|A| = -2$.

   (a) Is A invertible? *Explain your answer.*

   (b) Is there any $\boldsymbol{b} \in \mathbb{R}^4$ such that $A\boldsymbol{x} = \boldsymbol{b}$ does not have a solution? If so, give an example. If not, explain why.

   (c) Use Cramer's rule to compute x_4 such that $A \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 3 \end{pmatrix}$.

   Hint: By applying a column operation, you can make the 4×4 *determinant computation in this problem much easier.*

$* * *$ The exam continues on the next page. $* * *$

4. This question concerns the matrix $D = \begin{pmatrix} 1 & 1 \\ 1 & 0 \\ 1 & 1 \end{pmatrix}$ and the vector $\boldsymbol{b} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$.

   (a) Find **three** orthonormal vectors $\boldsymbol{q}_1, \boldsymbol{q}_2, \boldsymbol{q}_3$ in $\mathbb{R}^3$ such that $C(D) = \mathrm{Span}(\boldsymbol{q}_1, \boldsymbol{q}_2)$.

   (b) There is no solution to $D\boldsymbol{x} = \boldsymbol{b}$. Find the least squares approximate solution $\hat{\boldsymbol{x}}$.

   (c) What is the projection of $\boldsymbol{b}$ onto $C(D)$?

   (d) What is the projection of $\boldsymbol{b}$ onto $C(D)^{\perp}$?

5. (a) Suppose Q is an orthogonal $n \times n$ matrix. Is $\lambda = 0$ an eigenvalue of Q? *Explain your answer.*

   (b) Find the eigenvalues and eigenvectors of the matrix $\begin{pmatrix} 5 & 0 & 1 \\ 0 & 5 & 0 \\ 1 & 0 & 5 \end{pmatrix}$.

   (c) Let V be the subspace of $\mathbb{R}^4$ consisting of all vectors orthogonal to $\begin{pmatrix} 15\sqrt{2} \\ -520 \\ 2008 \\ \sqrt{17} \end{pmatrix}$.

   Let P be the 4×4 projection matrix onto V. Compute $\mathrm{tr}(P)$ and $\det(P)$, and explain your reasoning.

   Hint: Like the other parts of problem 5, (c) involves eigenvectors and eigenvalues.

$* * *$ This is the end of the exam. $* * *$