1. Compute each product of matrices, or indicate that the matrices are not compatible for multiplication:

(a) $\begin{pmatrix} 1 & 2 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ -3 \end{pmatrix}$

Solution: $\begin{pmatrix} 1 & 2 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ -3 \end{pmatrix} = 0$

(b) $\begin{pmatrix} 1 \\ 1 \\ -3 \end{pmatrix} \begin{pmatrix} 1 & 2 & 1 \end{pmatrix}$

Solution: $\begin{pmatrix} 1 \\ 1 \\ -3 \end{pmatrix} \begin{pmatrix} 1 & 2 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 1 \\ 1 & 2 & 1 \\ -3 & -6 & -3 \end{pmatrix}$

(c) $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 3 & 1 \\ 5 & 2 & 5 \end{pmatrix}$

Solution: $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 3 & 1 \\ 5 & 2 & 5 \end{pmatrix} = \begin{pmatrix} 5 & 2 & 5 \\ -1 & -3 & -1 \end{pmatrix}$

(d) $\begin{pmatrix} 1 & 3 & 1 \\ 5 & 2 & 5 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

Solution: Not compatible.

(e) $\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$

Solution: $\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}^7 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$.

This can be seen without any calculation, since this matrix is the row exchange matrix P_{13}. Switching rows seven times is the same as switching them once.

2. This question concerns the matrix

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & -1 \\ 2 & 1 & 1 \\ 0 & 1 & -1 \end{pmatrix}.$$

(a) Compute R, the row reduced echelon form of A.

Solution: Forward elimination:

$$\begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & -1 \\ 2 & 1 & 1 \\ 0 & 1 & -1 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \\ 0 & 1 & -1 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Backward elimination:

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

The pivots are 1, so there is no need to multiply the rows by scalars, and:

$$R = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

(b) What is the rank of A?

Solution: Since there are two pivots in R, we have $\text{Rank}(A) = 2$.

(c) Which columns are pivot columns, and which are free columns (if any)?

Solution:
Pivot columns: 1 and 2
Free column: 3

(d) Find the special solutions to $A\boldsymbol{x} = \mathbf{0}$.

Solution: The RREF matrix has block form $R = \begin{pmatrix} I & F \\ 0 & 0 \end{pmatrix}$, where $F = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$, so the null space matrix is

$$N = \begin{pmatrix} -F \\ I \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}.$$

Therefore there is one special solution,

$$\boldsymbol{x}_1 = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}.$$

(e) Compute the general solution to $A\boldsymbol{x} = \begin{pmatrix} 0 \\ 1 \\ -1 \\ 1 \end{pmatrix}$.

Solution: One way to find a particular solution is to notice that the given vector is -1 times the third column of A, so we can use

$$\boldsymbol{x}_p = \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix},$$

giving general solution

$$\boldsymbol{x} = \boldsymbol{x}_p + \boldsymbol{x}_{null} = \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix} + c \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}.$$

An alternate approach would be to perform the elimination steps above using the augmented matrix $[A\boldsymbol{b}]$ to obtain $[R\boldsymbol{c}]$, then solve $R\boldsymbol{x} = \boldsymbol{c}$ with $x_3 = 0$ (set free variables to zero). This gives

$$\boldsymbol{x} = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + c \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}.$$

Note that one obtains the same *set* of solutions by either method, but the value of c corresponding to a given solution depends on the choice of $\boldsymbol{x}_p$.

3. This question concerns the matrix

$$B = \begin{pmatrix} 1 & 2 & -5 \\ 0 & -1 & 2 \\ 4 & 10 & -23 \end{pmatrix}.$$

(a) Find an invertible square matrix E such that $EB = U$ is upper triangular.

(b) Compute the LU decomposition of B.

(c) Compute the inverse matrix B^{-1}.

Solution: It is easiest to solve all three parts at once using Gauss-Jordan elimination on $[BI]$**.**

$$\left(\begin{array}{ccc|ccc} \boxed{1} & 2 & -5 & 1 & 0 & 0 \\ 0 & -1 & 2 & 0 & 1 & 0 \\ 4 & 10 & -23 & 0 & 0 & 1 \end{array}\right) \qquad L = \begin{pmatrix} 1 & 0 & 0 \\ ? & 1 & 0 \\ ? & ? & 1 \end{pmatrix}$$

$$\longrightarrow \left(\begin{array}{ccc|ccc} \boxed{1} & 2 & -5 & 1 & 0 & 0 \\ 0 & \boxed{-1} & 2 & 0 & 1 & 0 \\ 0 & 2 & -3 & -4 & 0 & 1 \end{array}\right) \qquad L = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 4 & ? & 1 \end{pmatrix}$$

$$\longrightarrow \left(\begin{array}{ccc|ccc} \boxed{1} & 2 & -5 & 1 & 0 & 0 \\ 0 & \boxed{-1} & 2 & 0 & 1 & 0 \\ 0 & 0 & \boxed{1} & -4 & 2 & 1 \end{array}\right) \qquad L = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 4 & -2 & 1 \end{pmatrix}$$

So we have $\boxed{\text{(a) } E = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -4 & 2 & 1 \end{pmatrix}}$ $\boxed{\text{(b) } B = LU = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 4 & -2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & -5 \\ 0 & -1 & 2 \\ 0 & 0 & 1 \end{pmatrix}.}$

Note that in this case, E and L differ only in the signs of the entries below the diagonal. However, this is not always the case.

We continue with backward elimination and row scaling to find B^{-1}:

$$\left(\begin{array}{ccc|ccc} 1 & 2 & -5 & 1 & 0 & 0 \\ 0 & -1 & 2 & 0 & 1 & 0 \\ 0 & 0 & \boxed{1} & -4 & 2 & 1 \end{array}\right) \to \left(\begin{array}{ccc|ccc} 1 & 2 & 0 & -19 & 10 & 5 \\ 0 & \boxed{-1} & 0 & 8 & -3 & -2 \\ 0 & 0 & \boxed{1} & -4 & 2 & 1 \end{array}\right)$$

$$\to \left(\begin{array}{ccc|ccc} \boxed{1} & 0 & 0 & -3 & 4 & 1 \\ 0 & \boxed{-1} & 0 & 8 & -3 & -2 \\ 0 & 0 & \boxed{1} & -4 & 2 & 1 \end{array}\right)$$

$$\to \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -3 & 4 & 1 \\ 0 & 1 & 0 & -8 & 3 & 2 \\ 0 & 0 & 1 & -4 & 2 & 1 \end{array}\right)$$

So we have $\boxed{\text{(c) } B^{-1} = \begin{pmatrix} -3 & 4 & 1 \\ -8 & 3 & 2 \\ -4 & 2 & 1 \end{pmatrix}.}$

4. Let $\mathbf{M}_{2\times 3}$ denote the vector space of all 2×3 matrices.

(a) Let $V \subset \mathbf{M}_{2\times 3}$ denote the set of matrices whose third column is $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$.

Is V a subspace of $\mathbf{M}_{2\times 3}$? Either show that it is, or explain why it is not.

Solution: The sum of two elements of V also has zeros in the third column, because vector addition in $\mathbf{M}_{2\times 3}$ is defined elementwise. Therefore V is closed under vector addition.

Any multiple of an element of V has zeros in the third column, because scalar multiplication is defined elementwise. Therefore V is closed under scalar multiplication.

Since V is closed under vector addition and scalar multiplication, it is a subspace of $\mathbf{M}_{2\times 3}$.

(Note that V can also be described as the set of all 2×3 matrices A such that the null space of A contains $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$. One can show that this condition defines a subspace using the same method as the solution to (c) below.)

(b) Describe the span of the following vectors in $\mathbf{M}_{2\times 3}$:

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

(That is, describe it in a more useful way than "the span of the following vectors...".)

Solution: The most general linear combination of these matrices is

$$a \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} + b \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} a+b & a+b & a+b \\ a & a & a \end{pmatrix}$$

where a and b are any real numbers. We can write this as

$$\begin{pmatrix} c & c & c \\ a & a & a \end{pmatrix}$$

where $c = a + b$. Since a and b are arbitrary, a and c take on all real values. Thus the subspace spanned by the two matrices can be described as *the set of all 2×3 matrices whose columns are identical to one another.*

(c) Let $W \subset \mathbf{M}_{2\times 3}$ denote the set of matrices A such that the null space of A contains $\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$.

Show that W is a subspace of $\mathbf{M}_{2\times 3}$.

Solution: We must show that W is closed under vector addition and scalar multiplication. Suppose $A \in W$ and $c \in \mathbb{R}$. Then

$$(cA)\boldsymbol{v} = c(A\boldsymbol{v}) = c\mathbf{0} = \mathbf{0}$$

and so $cA \in W$.
Suppose $A, B \in W$. Then

$$(A+B)\boldsymbol{v} = A\boldsymbol{v} + B\boldsymbol{v} = \mathbf{0} + \mathbf{0} = \mathbf{0}$$

and so $(A+B) \in W$.
Therefore, W is a subspace of $\mathbf{M}_{2\times 3}$.

5. (a) What does it mean for a set of vectors $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_n$ to be *linearly dependent*? (Write a definition.)

Solution: The vectors are linearly dependent if there exist real numbers $c_1, c_2, \ldots, c_n \in \mathbb{R}$, not all zero, such that

$$c_1\boldsymbol{v}_1 + c_2\boldsymbol{v}_2 + \cdots + c_n\boldsymbol{v}_n = \mathbf{0}.$$

(b) Show (directly) that these four vectors are linearly dependent:

$$\begin{pmatrix}1\\1\\0\end{pmatrix}, \begin{pmatrix}1\\2\\2\end{pmatrix}, \begin{pmatrix}2\\0\\1\end{pmatrix}, \begin{pmatrix}1\\1\\2\end{pmatrix}$$

Solution: We must find a linear combination of these vectors that is equal to $\mathbf{0}$, or equivalently, an element of the null space of

$$A = \begin{pmatrix}1 & 1 & 2 & 1\\1 & 2 & 0 & 1\\0 & 2 & 1 & 2\end{pmatrix}.$$

Using elimination, we put A in echelon form.

$$\begin{pmatrix}1 & 1 & 2 & 1\\1 & 2 & 0 & 1\\0 & 2 & 1 & 2\end{pmatrix} \to \begin{pmatrix}1 & 1 & 2 & 1\\0 & 1 & -2 & 0\\0 & 2 & 1 & 2\end{pmatrix} \to \begin{pmatrix}1 & 1 & 2 & 1\\0 & 1 & -2 & 0\\0 & 0 & 5 & 2\end{pmatrix}$$

We see that the fourth column is free, and set $x_4 = 1$ to find a special solution by back-substitution.

$$5x_3 + 2(1) = 0 \quad\Rightarrow\quad x_3 = \frac{-2}{5}$$

$$x_2 - 2\left(\frac{-2}{5}\right) + 0(1) = 0 \quad\Rightarrow\quad x_2 = \frac{-4}{5}$$

$$x_1 + \frac{-4}{5} + 2\left(\frac{-2}{5}\right) + 1 = 0 \quad\Rightarrow\quad x_1 = \frac{3}{5}.$$

Therefore, we have

$$\begin{pmatrix}3/5\\-4/5\\-2/5\\1\end{pmatrix} \in N(A)$$

or equivalently (multiplying by 5),

$$3\begin{pmatrix}1\\1\\0\end{pmatrix} - 4\begin{pmatrix}1\\2\\2\end{pmatrix} - 2\begin{pmatrix}2\\0\\1\end{pmatrix} + 5\begin{pmatrix}1\\1\\2\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$$

(c) Explain why *any* four vectors in $\mathbb{R}^3$ are linearly dependent.

Solution: Put the four vectors into the columns of a 3×4 matrix A. The vectors are linearly dependent if there is a nonzero vector in the null space of A. However, any 3×4 matrix has at least one free column, and the associated special solution is a nonzero vector in the null space. Therefore, the four vectors are linearly dependent.