# Math 520 Exam 1 Topic Outline

Sections 1–3 (Xiao/Dumas/Liaw) · Spring 2008

*Exam 1 will be held on Tuesday, Feb 26, 7-8pm in 117 MacMillan*

**What will be covered**

The exam will cover material from the lectures up to and including Friday, February 22. This corresponds approximately to chapters 1 and 2, sections 3.1-3.4, and part of section 3.5 in Strang's book. There will be an emphasis on material *after* section 2.2.

**Outline of topics**

This outline is designed to help you plan your review for the first exam. It is not intended to cover every detail.

(1) Foundation in vector arithmetic (§§1.1 - 1.2)
   (a) Addition, scalar multiplication, the dot product, linear combinations

(2) Representing linear equations using matrices and vectors (§2.1)
   (a) A system of linear equations becomes $A\mathbf{x} = \mathbf{b}$ in matrix/vector notation.
   (b) *Row Picture* - each equation determines a subset (e.g. a plane in $\mathbb{R}^3$, a line in $\mathbb{R}^2$) of the space of potential solutions. Taking the intersection gives the actual solution set.
   (c) *Column Picture* - finding a solution $\mathbf{x}$ amounts to expressing the right hand side $\mathbf{b}$ as a linear combination of the columns of $A$.
   (d) Possible solution sets: *none at all*, *exactly one*, or *infinitely many.* If $\mathbf{x}$ and $\mathbf{y}$ are solutions, so is $\frac{1}{2}(\mathbf{x} + \mathbf{y})$.

(3) Solving Linear Systems $A\mathbf{x} = \mathbf{b}$ (§2.2 - §2.3)
   (a) *Gaussian Elimination* - row operations put an invertible square matrix $A$ in upper triangular form $U$, possibly after exchanging some rows.
   (b) The right hand side $\mathbf{b}$ is transformed into a vector $\mathbf{c}$ by applying the same row operations.
   (c) The *augmented matrix* $[A\ \mathbf{b}]$ is a convenient way to keep track of both the matrix and the right-hand side during elimination.
   (d) The *pivots* appear on the diagonal of $U$.
   (e) Back-substitution solves the triangular system $U\mathbf{x} = \mathbf{c}$ (and this is quite easy).
   (f) Each row operation in the elimination process can be represented by an elimination matrix $E_{ij}(\ell)$. Its looks like the $n \times n$ *identity matrix* except for the $(i, j)$ entry, which is $-\ell$.
   (g) To determine the matrix representing a row operation (subtraction or row exchange), apply the same operation to the identity matrix.

(4) Matrix Algebra (§§2.4-2.5)
   (a) If $A$ is $m \times n$ and $B$ is $n \times p$ then the product $AB$ exists and is an $m \times p$ matrix.
   (b) Each entry of $AB$ is the dot product of a *row from $A$* and a *column from $B$*.
   (c) The summation formula

$$(AB)_{ij} = \sum_{k=1}^{n} A_{ik} B_{kj}$$

   (d) Matrix multiplication is associative ($(AB)C = A(BC)$) but it is not commutative (usually $AB \neq BA$).
   (e) The square *identity matrix* $I$ has 1 on the diagonal and 0 elsewhere.
   (f) The *inverse* of a square matrix $A$ is another matrix $A^{-1}$ such that

$$AA^{-1} = A^{-1}A = I$$

   (g) The inverse of a product: $(AB)^{-1} = B^{-1}A^{-1}$
   (h) The *transpose* operation: if $A$ is $m \times n$, then $A^T$ is $n \times m$.

$$(A^T)_{ij} = A_{ji}$$

   (i) The transpose of a product: $(AB)^T = B^T A^T$
   (j) The inverse and transpose can be interchanged: $(A^T)^{-1} = (A^{-1})^T$
   (k) A permutation matrix $P$ has a exactly one 1 in each row and each column; all other entries are zero.
   (l) Permutation matrices satisfy $P^T = P^{-1}$.

(5) Finding the Inverse (§2.5)
   (a) A square matrix may or may not have an inverse.
   (b) The following are equivalent for a $n \times n$ (square) matrix:
      (i) $A$ has an inverse
      (ii) $A$ has $n$ pivots (which are nonzero)
      (iii) $A\mathbf{x} = \mathbf{b}$ has a solution for every $\mathbf{b}$
      (iv) $A\mathbf{x} = \mathbf{0}$ only when $\mathbf{x} = \mathbf{0}$
   (c) The definition of the inverse $AA^{-1} = I$ is a system of linear equations for the entries of $A^{-1}$.
   (d) Gauss-Jordan elimination solves these equations simultaneously; it transforms the augmented matrix $[A\ I]$ into $[I\ A^{-1}]$ using elimination (downward and upward) and dividing rows by pivots.

(6) Elimination as Factorization (§§2.6-2.7)
   (a) Gaussian elimination without row exchanges allows one to express an invertible matrix $A$ as a product of a lower triangular matrix $L$ and an upper triangular matrix $U$:

$$A = LU$$

   (b) The matrix $U$ is the final result of Gaussian elimination, while $L$ is a table of the multipliers used during elimination.
   (c) In the $LU$ decomposition, $L$ has 1 on the diagonal, while $U$ has pivots on the diagonal.
   (d) The closely related $LDU$ decomposition separates the pivots from the upper triangular matrix $U$. The diagonal matrix $D$ contains the pivots, while the new upper triangular matrix $U'$ is obtained from $U$ by dividing each row by its pivot.

$$A = LDU'$$

   (e) In the $LDU$ decomposition, both $L$ and $U$ have 1s on the diagonal.
   (f) The $LU$ decomposition gives a new way to solve $A\mathbf{x} = \mathbf{b}$: first solve $L\mathbf{c} = \mathbf{b}$, then solve $U\mathbf{x} = \mathbf{c}$. Both are triangular systems so this is easy, using substitution.
   (g) If row exchanges are necessary, the corresponding statement is that $PA$ has an $LU$ decomposition for some permutation matrix $P$:

$$PA = LU$$

(7) Vector Spaces and Subspaces (§3.1)
  (a) A *vector space* is a set $V$ together with two operations, *vector addition* and *scalar multiplication*, satisfying the rules on page 118 of Strang.
  (b) Examples of vector spaces: $\mathbb{R}^n$, $\mathbf{M}_{2\times 2}$, $\mathbf{F}$, $\mathbf{Z}$
  (c) A *subspace* is a subset of a vector space that is *closed* under vector addition and scalar multiplication.
  (d) A subset $W \subset V$ is a subspace if and only if both of these conditions hold:
    (i) For all $\mathbf{x}, \mathbf{y} \in W$, the sum $(\mathbf{x}+\mathbf{y})$ is also in $W$
    (ii) For all $\mathbf{x} \in W$ and $c \in \mathbb{R}$, the product $c\mathbf{x}$ is in $W$
  (e) All subspaces contain the zero vector $\mathbf{0}$.
  (f) If a subspace contains $\mathbf{x}$, then it contains the entire line of multiples of $\mathbf{x}$, i.e. $\{c\mathbf{x} \mid c \in \mathbb{R}\}$.
  (g) If a subspace contains a certain set of vectors, then it also contains any linear combination of those vectors.
  (h) The zero vector by itself is a subspace of any vector space.
  (i) Subspaces of $\mathbb{R}^2$:
    (i) The zero vector $\mathbf{0}$
    (ii) A line containing $\mathbf{0}$
    (iii) The entire plane $\mathbb{R}^2$
  (j) Subspaces of $\mathbb{R}^3$:
    (i) The zero vector $\mathbf{0}$
    (ii) A line containing $\mathbf{0}$
    (iii) A plane containing $\mathbf{0}$
    (iv) The entire space $\mathbb{R}^3$

(8) Column Space and Null Space (§§3.1-3.2)
  (a) The *column space* $C(A)$ of an $m \times n$ matrix $A$ is the subspace of $\mathbb{R}^m$ consisting of all linear combinations of the columns of $A$. (There are $n$ columns, each has $m$ entries.)
  (b) A system of linear equations $A\mathbf{x} = \mathbf{b}$ has a solution if and only if $\mathbf{b} \in C(A)$.
  (c) The *null space* $N(A)$ of an $m \times n$ matrix $A$ is the set of all vectors $\mathbf{x}$ such that $A\mathbf{x} = \mathbf{0}$. It is a subspace of $\mathbb{R}^n$.
  (d) $N(A)$ is a subspace because $A$ is linear, i.e.
    (i) $A(\mathbf{x}+\mathbf{y}) = A\mathbf{x} + A\mathbf{y}$
    (ii) $A(c\mathbf{x}) = c(A\mathbf{x})$
  (e) For a square invertible matrix, $N(A) = \mathbf{0}$, i.e. the only solution to $A\mathbf{x} = \mathbf{0}$ is $\mathbf{x} = \mathbf{0}$.
  (f) If $\mathbf{x}_{particular}$ is a solution to $A\mathbf{x} = \mathbf{b}$, and $\mathbf{x}_{null} \in N(A)$, then $(\mathbf{x}_{particular} + \mathbf{x}_{null})$ is another solution of $A\mathbf{x} = \mathbf{b}$.
  (g) Finding the null space means solving $A\mathbf{x} = \mathbf{0}$; to do this, put $A$ in *echelon form*, obtaining a matrix $U$.
  (h) The number of pivots obtained by putting $A$ in echelon form is the *rank* of $A$ (which we call $r$).
  (i) Solving $U\mathbf{x} = \mathbf{0}$ is equivalent to solving $A\mathbf{x} = \mathbf{0}$.
  (j) The echelon matrix $U$ has $r$ *pivot columns* and $(n-r)$ *free columns*. There are corresponding *pivot variables* and *free variables*.
  (k) The free columns of $U$ are linear combinations of the previous columns. The pivot columns are *not* linear combinations of previous columns.
  (l) To find vectors in the null space, assign one of the free variables the value 1, and set the rest to 0. Solve for the pivot variables to get a *special solution*.

(m) The null space of $A$ consists of the linear combinations of the $n-r$ special solutions.

(9) The Reduced Row Echelon Form (§3.3)
(a) An $m \times n$ matrix can be put into *reduced row echelon form* (RREF) where each pivot column has a single nonzero entry, and this entry is 1.
(b) To get the RREF $R$ from the echelon form $U$, eliminate upwards to clear the entries above the pivots, then divide each row by its pivot.
(c) The RREF of a matrix is unique.
(d) The RREF matrix $R$ makes it easier to find the special solutions, because the coefficients of the free columns of $R$ appear in the pivot variables with opposite sign.
(e) The special solutions can be collected into the columns of a *null space matrix* $N$. It has size $n \times (n-r)$.

(10) The complete solution to $A\mathbf{x} = \mathbf{b}$ (§3.4)
(a) The rank of an $m \times n$ matrix is the number of pivots. The rank $r$ satisfies $r \leq m$ and $r \leq n$.
(b) The RREF $R$ has $m - r$ zero rows.
(c) To decide whether or not $A\mathbf{x} = \mathbf{b}$ has any solutions, reduce the augmented matrix $[A\ \mathbf{b}]$ to $[R\ \mathbf{c}]$. There is a solution if and only if the last $m - r$ entries of $\mathbf{c}$ are zero (so the last $m - r$ equations become $0 = 0$).
(d) To find a *particular solution* $\mathbf{x}_p$, set all $n-r$ free variables to zero and solve $R\mathbf{x} = \mathbf{c}$ by back-substitution.
(e) The *general solution* is then $\mathbf{x} = \mathbf{x}_p + \mathbf{x}_n$ where $\mathbf{x}_n \in N(A)$, i.e. $\mathbf{x}_n$ is a linear combination of the $(n-r)$ special solutions.
(f) Special case: If $r = n$, then $A$ has *full column rank*. There is exactly one solution if $\mathbf{b} \in C(A)$ and no solution otherwise. (In particular, if there is any solution at all, it is unique.)
(g) Special case: If $r = m$ then $A$ has *full row rank*. There is exactly one solution if $m = n$, and infinitely many solutions if $m < n$. (In particular, there is at least one solution for any $\mathbf{b}$.)
(h) Special case: If $r = m = n$ then $A$ is a *square invertible matrix*. There is exactly one solution for every $\mathbf{b}$. In fact, the solution is $\mathbf{x} = A^{-1}\mathbf{b}$.

(11) Linear Independence, Span, Basis, and Dimension (§3.5)
(a) Vectors $\mathbf{x}_1, \ldots, \mathbf{x}_n$ are *linearly independent* if

$$c_1\mathbf{x}_1 + \ldots + c_n\mathbf{x}_n \neq \mathbf{0}$$

except when $c_1 = c_2 = \cdots = c_n = 0$.
(b) The columns of a matrix $A$ are linearly independent exactly when $\mathbf{x} = \mathbf{0}$ is the only solution to $A\mathbf{x} = 0$, i.e. $N(A) = \mathbf{0}$.
(c) The *span* of $\mathbf{x}_1, \ldots, \mathbf{x}_n$ is the vector space consisting of all linear combinations of these vectors.
(d) We say that $\mathbf{x}_1, \ldots, \mathbf{x}_n$ *span* $V$ if $V$ is equal to the span of $\mathbf{x}_1, \ldots, \mathbf{x}_n$.
(e) A *basis* of $V$ is a set of vectors that span $V$ and are linearly independent.
(f) Every basis of a vector space has the same number of elements. This number is the *dimension* of $V$, written $\dim V$.
(g) If $\dim V = d$, then any $d$ linearly independent vectors form a basis of $V$.
(h) If $\dim V = d$, then $(d-1)$ or fewer vectors cannot span $V$, and $(d+1)$ or more vectors cannot be linearly independent.
(i) The columns of an $n \times n$ matrix $A$ are a basis for $\mathbb{R}^n$ if and only if $A$ is invertible.