1. Let $f(x, y) = x^3 + y^3 - 3x - 3y + 9$.

   (a) Compute the gradient $\nabla f(x, y)$.

   Solution:
$$\nabla f(x,y) = \langle 3x^2 - 3,\ 3y^2 - 3 \rangle$$

   (b) Find and classify the critical points of f.

   Solution: The critical points are the solutions to $\nabla f = \mathbf{0}$, i.e. to the pair of equations $3x^2 - 3 = 0$ and $3y^2 - 3 = 0$. The solutions are $x = \pm 1$ and $y = \pm 1$ so there are four critical points: $(1, 1)$, $(1, -1)$, $(-1, 1)$, $(-1, -1)$.

   Now we apply the second derivative test. Note that $f_{xx} = 6x$ and
$$D = \begin{vmatrix} 6x & 0 \\ 0 & 6y \end{vmatrix} = 36xy$$
   so we have:

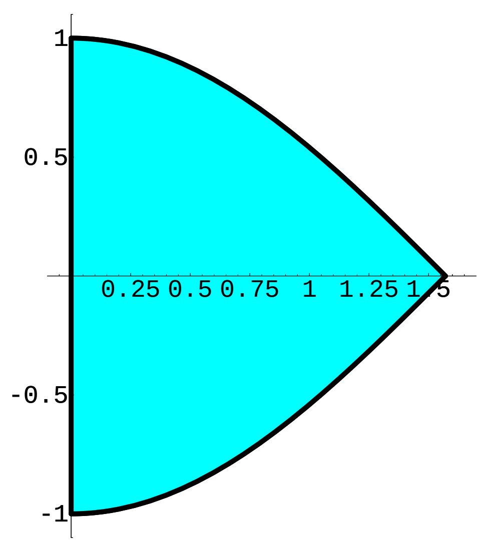
Point	D	f_{xx}	Type
$(1, 1)$	36	6	local minimum
$(1, -1)$	-36	$*$	saddle
$(-1, 1)$	-36	$*$	saddle
$(-1, -1)$	36	-6	local maximum

   The entries marked "$*$" in the table are not necessary for the classification, since $D < 0$ indicates a saddle point regardless of the value of f_{xx}.

2. Let D be the type I region in $\mathbb{R}^2$ between the graphs of $y = \cos(x)$ and $y = -\cos(x)$ for $0 \le x \le \frac{\pi}{2}$.

   (a) Sketch the region D.

   Solution:



(b) Write an iterated integral that represents the area of D, and then compute this area.

Solution:

$$\text{Area} = \iint_D 1\, dA = \int_0^{\frac{\pi}{2}} \int_{-\cos(x)}^{\cos(x)} dydx = \int_0^{\frac{\pi}{2}} 2\cos(x)\, dx = 2\left(\sin(\frac{\pi}{2}) - \sin(0)\right) = 2$$

(c) Find the centroid of D.

Solution: Because the region D is symmetric with respect to reflection in the x axis, the centroid lies on the x axis, i.e. $\overline{y} = 0$. Thus we need only compute $\overline{x}$:

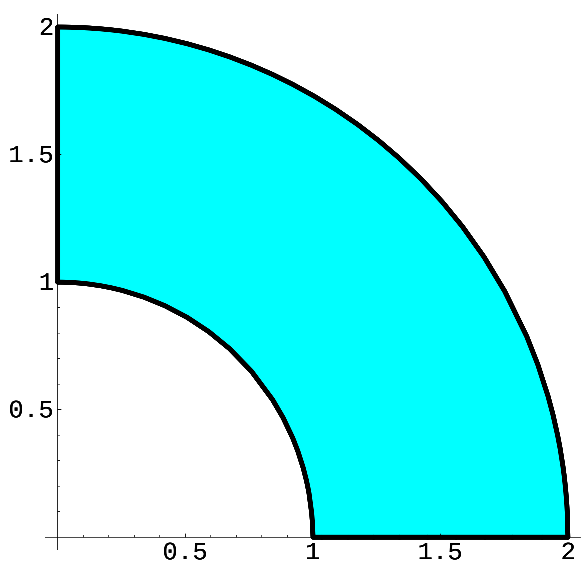
$$\begin{aligned}\overline{x} = \frac{1}{\text{Area}(D)} \iint_D x dA &= \frac{1}{2} \int_0^{\frac{\pi}{2}} \int_{-\cos(x)}^{\cos(x)} x\, dydx \\ &= \int_0^{\frac{\pi}{2}} x\cos(x)\, dx = [x\sin(x) + \cos(x)]_0^{\frac{\pi}{2}} = \frac{\pi}{2} - 1\end{aligned}$$

Therefore the centroid is $\left(\frac{\pi}{2} - 1, 0\right)$.

3. Let Q be the region in the first quadrant of $\mathbb{R}^2$ between the circles $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$.

(a) Sketch the region Q.

Solution:



(b) Let $g(x, y) = \dfrac{x\, e^{x^2+y^2}}{\sqrt{x^2 + y^2}}$. Compute $\displaystyle\iint_Q g(x, y)\, dA$.

Solution: The region Q is the polar rectangle $\{(r,\theta) \mid 1 \le r \le 2,\ 0 \le \theta \le \frac{\pi}{2}\}$, and we have

$$g(r\cos(\theta), r\sin(\theta)) = e^{r^2}\cos(\theta),$$

therefore

$$\begin{aligned}\iint_Q g(x,y)\,dA &= \int_0^{\frac{\pi}{2}}\int_1^2 re^{r^2}\cos(\theta)\,drd\theta \\ &= \left(\int_0^{\pi/2}\cos(\theta)\,d\theta\right)\left(\int_1^2 re^{r^2}\,dr\right) = \frac{e^4-e}{2}\end{aligned}$$

4. Let T be the solid tetrahedron in $\mathbb{R}^3$ bounded by the three coordinate planes and the plane $x+y+z=1$.

   (a) Compute the volume of T.

Solution: We use a triple integral, viewing T as a z-simple region over a triangle in the xy plane:

$$\text{Volume} = \iiint_T 1\,dV = \int_0^1\int_0^{1-x}\int_0^{1-x-y} dzdydx = \int_0^1\int_0^{1-x}(1-x-y)\,dydx$$
$$= \int_0^1 \frac{1}{2}(1-x)^2\,dx = \frac{1}{6}$$

   (b) Suppose that T represents an object whose density is given by $\rho(x,y,z)=z$. Compute the mass of the object.

Solution:

$$m = \iiint_T \rho(x,y,z)\,dV = \int_0^1\int_0^{1-x}\int_0^{1-x-y} z\,dzdydx = \int_0^1\int_0^{1-x}\frac{1}{2}(1-x-y)^2\,dydx$$
$$= \int_0^1 \frac{1}{6}(1-x)^3\,dx = \frac{1}{24}$$

5. Compute the volume of the solid ellipsoid

$$E = \{(x,y,z) \mid \frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2} \le 1\}$$

where $a,b,c>0$.

Solution: Let $T(u, v, w) = (au, bv, cw)$. This is an injective C^1 transformation and $T(B) = E$, where B is the closed unit ball in uvw-space. Its Jacobian is

$$\frac{\partial(x, y, z)}{\partial(u, v, w)} = \begin{vmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{vmatrix} = abc > 0.$$

Therefore

$$\text{Volume}(E) = \iiint_E dV = \iiint_B abc\, dudvdw = (abc)\text{Volume}(B) = \frac{4}{3}\pi abc.$$

In the last step we used the standard formula $V = \frac{4}{3}\pi r^3$ for the volume of a ball of radius r in $\mathbb{R}^3$. Of course we could also compute the volume of B directly by integrating in spherical coordinates.