1. Find the line containing the origin in $\mathbb{R}^3$ that perpendicularly intersects the line

$$\mathbf{l}(t) = \langle 3, 5, 1\rangle + t\langle 2, 4, 3\rangle.$$

Solution: Let $\langle a, b, c\rangle$ be the position vector of the point where the lines meet. Since this point lies on the given line, we have

$$\langle a, b, c\rangle = \langle 3 + 2t, 5 + 4t, 1 + 3t\rangle$$

for some t. Since the lines are orthogonal, we have

$$\langle a, b, c\rangle \cdot \langle 2, 4, 3\rangle = 0.$$

Combing these,

$$0 = 2(3 + 2t) + 4(5 + 4t) + 3(1 + 3t) = 29 + 29t$$

so $t = -1$ and $\langle a, b, c\rangle = \langle 3 - 2, 5 - 4, 1 - 3\rangle = \langle 1, 1, -2\rangle$. Therefore the orthogonal line has vector equation

$$\mathbf{r}(t) = t\langle 1, 1, -2\rangle$$

or symmetric equation

$$x = y = -2z.$$

2. Consider the parameterized curve $\mathbf{r}(t) = \langle t^2, \frac{1}{3}t^3, -t^2\rangle$.

(a) Find the unit tangent vector at $t = 2$.

Solution: The derivative $\mathbf{r}'(t) = \langle 2t, t^2, -2t\rangle$ has norm $|\mathbf{r}'(t)| = \sqrt{8t^2 + t^4}$, so the unit tangent vector is

$$\mathbf{T}(2) = \frac{\mathbf{r}'(2)}{|\mathbf{r}'(2)|} = \frac{1}{4\sqrt{3}}\langle 4, 4, -4\rangle = \frac{1}{\sqrt{3}}\langle 1, 1, -1\rangle.$$

(b) Find the arc length of the curve between $t = 0$ and $t = 1$.

Solution: The length is

$$\begin{aligned}\int_0^1 |\mathbf{r}'(t)|\,dt &= \int_0^1 t\sqrt{8 + t^2}\,dt \\ &= \left[\frac{1}{3}\left(8 + t^2\right)^{\frac{3}{2}}\right]_0^1 \\ &= 9 - \frac{16\sqrt{2}}{3}\end{aligned}$$

3. Let $f(x, y) = e^{-x^2-2y^2}$.

(a) Find the domain of f.

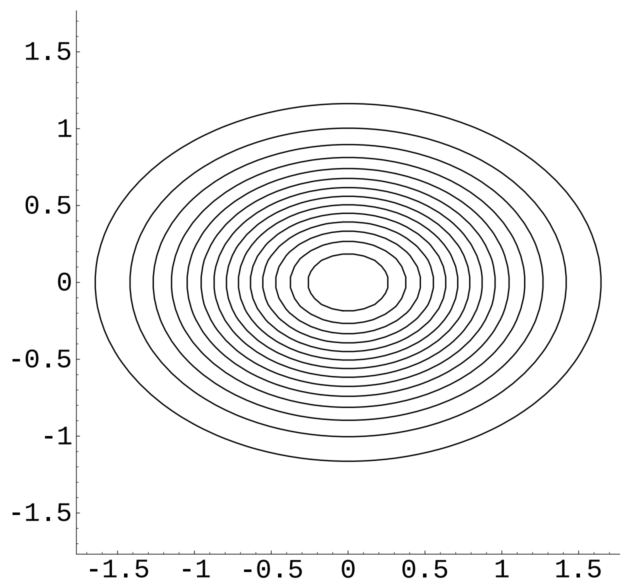
Solution: The function is well-defined for all values of x and y, so the domain is $\mathbb{R}^2$.

(b) Find the range of f.

Solution: The polynomial $p(x,y) = -x^2 - 2y^2$ can take on any negative real value and zero. As $u \to -\infty$, $e^u \to 0$, and $e^0 = 1$, so the range of $f(x,y) = e^{p(x,y)}$ is $(0,1]$.

(c) Draw a contour plot of the function f showing several level curves.

Solution: For $0 < k < 1$, the contour $e^{-x^2-2y^2} = k$ is the curve $x^2 + 2y^2 = -\log k$. Note that $\log k < 0$ since $k < 1$. Thus each contour is an ellipse centered at the origin, with the x axis as its major axis and the y axis as its minor axis.
The contours with $k = \frac{1}{15}, \frac{2}{15}, \cdots, \frac{14}{15}$ are shown below.



4. Find the limit or show that it does not exist.

(a) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2}{x^2+y^2}$

Solution: Let $g(x,y) = \frac{x^2}{x^2+y^2}$. Along the x axis we have $g(x,0) = 1 \to 1$ as $x \to 0$, while along the y axis $g(0,y) = 0 \to 0$ as $y \to 0$. Thus the limit does not exist.

(b) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{xy(x-y)}{x^3+y^3+(x-y)^3}$

Solution: Let $h(x,y) = \frac{xy(x-y)}{x^3+y^3+(x-y)^3}$. Along the x axis we have $h(x,0) = 0 \to 0$ as $x \to 0$. The same behavior is observed along the y axis and the line $x = y$. But along

$y = 2x$ we have

$$f(x, 2x) = \frac{-2x^3}{x^3 + 8x^3 - x^3} = -\frac{1}{4} \to -\frac{1}{4} \quad \text{as} \quad x \to 0$$

so the limit does not exist.

5. Let $F(x, y) = \sqrt{4 - x^2 - 2y^2}$.

(a) Compute the partial derivatives $\dfrac{\partial F}{\partial x}$ and $\dfrac{\partial F}{\partial y}$.

Solution:

$$\frac{\partial F}{\partial x} = \frac{-x}{\sqrt{4 - x^2 - 2y^2}}$$

$$\frac{\partial F}{\partial x} = \frac{-2y}{\sqrt{4 - x^2 - 2y^2}}$$

(b) Find an equation of the tangent plane to the surface $z = F(x, y)$ at the point $(1, -1, 1)$.

Solution: We use the general equation

$$z - z_0 = F_x(x_0, y_0)\,(x - x_0) + F_y(x_0, y_0)\,(y - y_0)$$

substituting $F_x(1, -1) = -1$ and $F_y(1, -1) = 2$ to obtain

$$z - 1 = -(x - 1) + 2(y + 1)$$

or

$$x - 2y + z - 4 = 0.$$

(c) Is F differentiable at $(1, -1)$?

Solution: The partial derivatives F_x and F_y exist and are continuous near $(1, -1)$, so F is differentiable there. (In fact, F, F_x, and F_y are all continuous on the domain $\{(x, y) \mid x^2 + 2y^2 < 4\}$.)