Math 104 Homework 9

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Due Thursday, April 20, 2006

(1) **A regular 4-gon.**

(a) In the Minkowski model, find unit timelike vectors representing the vertices of a regular (equilateral, equiangular) quadrilateral Q centered at $O = (0, 0, 1)$ with all internal angles $\pi/4$. (Hint: for some value of t, the vectors $(\pm 1, 0, t)$, $(0, \pm 1, t)$ are spacelike and represent the lines containing the four edges of Q. Use the angle condition to find the right value of t, then find the vertices.)

(b) Compute the side length of Q.

(c) Compute the radius of the inscribed circle of Q.

(d) In the Poincaré model, Q is represented by four circular arcs. Find the centers and radii of the circles. (Hint: The pole of a Klein model line is the center of the corresponding Poincaré circle.)

(e) By decomposing Q into triangles, compute its area.

(2) **Beachfront property.**

(a) Let D be an open disk of radius R in the Euclidean plane. What fraction of the area of D consists of points within distance 1 of the boundary circle? (That is, let D_0 denote the set of points in D whose distance from the center O is at least $(R - 1)$. Compute $\operatorname{Area}(D_0)/\operatorname{Area}(D)$.)

(b) Let $b_E(R)$ denote the ratio from (a). Compute $\lim_{R\to\infty} b_E(R)$.

(c) Now let D be an open disk of radius R in the hyperbolic plane. What fraction of the area of D consists of points within distance 1 of the boundary circle?

(d) Let $b_H(R)$ denote the ratio from (c). Compute $\lim_{R\to\infty} b_H(R)$.

Note: The result of (d) is an example of the following principle: "On an island in the hyperbolic plane, most property is beachfront property."

(3) **Horocycles.**

(a) Let A and B be distinct points in $\mathbb{H}^2$. Show that there are exactly two horocycles containing both A and B, and that these are related by reflection in the line $\overline{AB}$. (Hint: First use reflections to reduce to the case where A and B lie on the x axis in Δ_P with midpoint O.)

(b) Let h be a horocycle containing A and B, and let $\tilde{h}$ denote the arc of h "between" A and B (as in Figure 1).

Let $\lambda(d)$ denote the length of $\tilde{h}$, which depends only on the distance $d = d(A, B)$ (you do not need to prove this). Compute $\lambda(d)$.

(c) Based on your formula from part (b), is following a horocycle an efficient way to get from A to B? Justify your answer by computing $\lim_{d\to\infty} \frac{\lambda(d)}{d}$.

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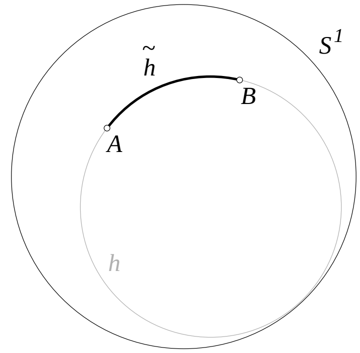

FIGURE 1. A path $\tilde{h}$ joining A and B along a horocycle.

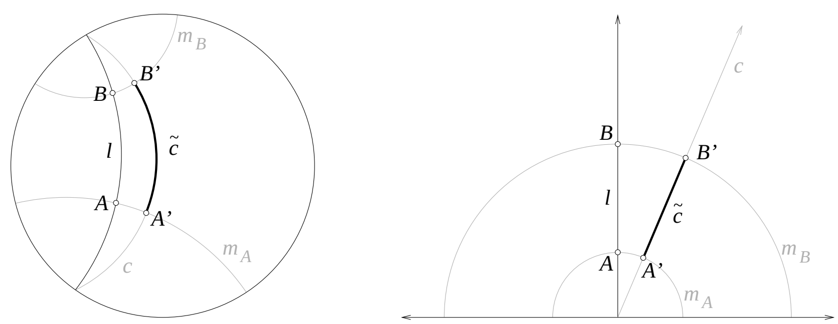

FIGURE 2. A hypercycle path between A' and B' in the Poincaré disk (left) and upper half-plane (right) models.

(4) **Hypercycles.** Let A and B be distinct points on a line l in $\mathbb{H}^2$. Let m_A and m_B denote the perpendiculars to l at A and B, respectively. Choose one side of l, and let A' and B' denote the points on m_A and m_B on the chosen side of l whose distances from A and B are h, respecitvely.

Then A' and B' are joined by an arc $\tilde{c}$ of an h-equidistant hypercycle c of l (as in Figure 2).

(a) Compute the length $\mu(d,h)$ of $\tilde{c}$, which depends only on $d = d(A,B)$ and h. (Hint: Work in the upper half plane model, let $A = (0,1)$ and $B = (0,e^d)$. Use the distance-angle formula from homework 8.)

(b) The calculation from (a) can be interpreted as follows: Suppose two points P and Q maintain a constant distance h while one of them, say P, moves distance d along a geodesic perpendicular to their separation. Then the ratio of the distance traveled by P to that traveled by Q is $\frac{\mu(d,h)}{d}$.
Describe the limiting behavior of this ratio as a function of d and h (computing, at the very least, $\lim_{h\to\infty} \frac{\mu(d,h)}{d}$).

(c) Compute the corresponding ratio $\frac{\mu(d,h)}{d}$ in Euclidean geometry. Comment on the difference between the hyperbolic and Euclidean situations.