Math 104 Homework 8

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Due Thursday, April 13, 2006

(1) **Distance formula for hypercycles.** In class we showed that the d-equidistant locus of the line in H represented by the positive y axis is a pair of Euclidean rays making angle θ with the y axis. Show that $\tan(\theta) = \sinh(d)$. (Hint: See problem (3) from homework 7.)

(2) **Collinearity.**
   (a) Let c be one of the following objects in the hyperbolic plane:
   - A horocycle
   - A hypercycle
   - A circle

   Show that no three points on c are collinear. (Note: Three points are collinear if they lie on a hyperbolic geodesic. This has nothing to do with the Euclidean lines in a model of $\mathbb{H}^2$.)

   (b) What is the relationship between (a) and Theorem 6.3 in Greenberg?

(3) **Minkowski space.** Let $v, w \in \mathbb{R}^{2,1}$. Decide whether each of the following statements is true or false. If it is true, prove it. If it is false, give a counterexample.
   (a) If v and w are null, then $\langle v, w \rangle = 0$.
   (b) If v and w are timelike, then $\langle v, w \rangle < 0$.
   (c) If v and w are spacelike, then $\langle v, w \rangle > 0$.
   (d) If v is null, then $v \in v^\perp$.
   (e) If $v \in v^\perp$, then v is null.
   (f) If v and w are timelike, then any linear combination of v and w is timelike.
   (g) If v and w are spacelike, then $\frac{v+w}{2}$ is spacelike.

(4) **Minkowski model distance.** Let $v_1 = (-1, 0, 2)$, $v_2 = (0, -1, 2)$, $w_1 = (2, 0, 1)$, $w_2 = (0, 2, 1)$. Note that v_1 and v_2 are timelike, so they represent points in $\mathbb{H}^2$, while w_1 and w_2 are spacelike, so they represent lines in $\mathbb{H}^2$. (Note: Since v_i are not unit vectors, they must be rescaled to lie on Σ.) Use the Minkowski inner product to compute the following hyperbolic quantities:
   (a) The distance between v_1 and v_2
   (b) Cancelled. (Was: The perpendicular distance from v_1 to the line w_1)
   (c) Cancelled. (Was: The perpendicular distance from v_1 to the line w_2)
   (d) The angle between the lines w_1 and w_2
   (e) Finally, draw a picture of the points v_i and lines w_i in the Klein model.

(5) **Klein distance and Minkowski distance.**
   (a) Find the unit vector in $\mathbb{R}^{2,1}$ that corresponds to the point $(r, 0) \in \Delta_K$.
   (b) Suppose that $x > 1$. Show that the unique positive solution of the equation $\cosh(d) = x$ is
   $$d = \log(x + \sqrt{x^2 - 1}).$$
   (c) Use the Minkowski inner product to compute the distance between the points in Σ corresponding to $(0, 0)$ and $(r, 0)$ in Δ_K. Simplify the resulting expression to show that it agrees with the Klein model distance $\frac{1}{2} \log \frac{1+r}{1-r}$.