Math 104 Homework 7

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(1) **Triangles and circles.** An *open disk* is the set of all points inside a circle.

(a) Prove that in the Euclidean plane, every open disk is contained in the interior of some triangle.

The rest of this exercise guides you through a proof that the corresponding statement in hyperbolic geometry is false, and in fact a disk contained in a triangle in $\mathbb{H}^2$ is necessarily quite small. This means that hyperbolic triangles are "skinny".

(b) Let T be a triangle in the hyperbolic plane. Show that there is an ideal triangle $\hat{T}$ such that T lies in the interior of $\hat{T}$. (Hint: Use the Klein model.)

(c) Suppose $\hat{T}$ is an ideal triangle in Δ_K with ideal vertices $P, Q, R \in S^1$ and that $O = (0,0)$ lies in the interior of $\hat{T}$. Show that one of the angles $\measuredangle POQ$, $\measuredangle QOR$, $\measuredangle ROP$ is less than or equal to $\frac{2\pi}{3}$. (Note: Euclidean angles and hyperbolic angles agree at $O \in \Delta_K$, so this is really a Euclidean question.)

(d) Let $P, Q \in S^1$ be ideal points that do not lie on a diameter, and suppose $|\measuredangle POQ| = \theta$. Show that the perpendicular distance $d(\theta)$ from O to l_{PQ} is a decreasing function of θ.

For the rest of this exercise, fix a real number d_0 such that $d_0 > d\left(\frac{2\pi}{3}\right)$, where $d(\theta)$ is the function from part (d).

(e) Let D be the open disk in the Klein model with center O and hyperbolic radius d_0. Use the results from (b)-(d) to show that D is not contained in the interior of any hyperbolic triangle.

(f) Let T be a hyperbolic triangle and P a point in the interior of T. Show that the open disk with center P and radius d_0 is not contained in the interior of T.

(2) **Distance in H.** There is a nice formula for the distance function in the upper half-plane model H that does not involve cross-ratios. Let $A = (x_A, y_A)$ and $B = (x_B, y_B)$ be points in H; define

$$\delta(A, B) = 1 + \frac{\|A - B\|^2}{2 y_A y_B}$$

where as usual

$$\|A - B\|^2 = (x_A - x_B)^2 + (y_A - y_B)^2.$$

The goal of this problem is to prove that

$$\cosh(d_H(A, B)) = \delta(A, B).$$

(a) Show that δ is invariant under horizontal translation and dilation of $\mathbb{R}^2$ centered at $(0,0)$. In other words, if A', B' are obtained from A, B by applying either a horizontal translation or a dilation centered at $(0,0)$, then

$$\delta(A', B') = \delta(A, B).$$

(b) Use (a) to show that if

$$\cosh(d_H(A, B)) = \delta(A, B)$$

for all A and B such that $\|A\| = \|B\| = 1$, then the formula actually holds for any A and B that do not lie on a vertical line.

(c) Show that $\cosh(d_H(A,B)) = \delta(A,B)$ if A and B lie on the same vertical line (i.e. $x_A = x_B$).

(d) Suppose $\|A\| = \|B\| = 1$ and $x_B > x_A$. Show that

$$e^{d_H(A,B)} = \sqrt{\frac{(1-x_A)(1+x_B)}{(1+x_A)(1-x_B)}}.$$

(e) Use (d) to show that $\cosh(d_H(A,B)) = \delta(A,B)$ if $\|A\| = \|B\| = 1$. (Combined with (b) and (c), this completes the proof of the formula.)

(3) **Parameterized lines.** Let l be a line in $\mathbb{H}^2$. A *parameterization of l by arc length* is a map $\gamma : \mathbb{R} \to \mathbb{H}^2$ such that $\gamma(t) \in l$ for all $t \in \mathbb{R}$, and such that the the distance from $\gamma(t_1)$ to $\gamma(t_2)$ is $|t_2 - t_1|$.

(a) Suppose $\gamma : \mathbb{R} \to \mathbb{H}^2$ satisfies all of the following:

(i) $\gamma(t) \in l$ for all $t \in \mathbb{R}$

(ii) $\gamma(\mathbb{R}^+)$ and $\gamma(\mathbb{R}^-)$ are opposite rays based at $\gamma(0)$, where $\mathbb{R}^+ = \{x \in \mathbb{R} \,|\, x \geq 0\}$ and $\mathbb{R}^- = \{x \in \mathbb{R} \mid x \leq 0\}$.

(iii) $d(\gamma(0), \gamma(t)) = |t|$ for all $t \in \mathbb{R}$

Show that γ is a line parameterized by arc length.

(b) Show that $\gamma(t) = (\tanh(d/2), 0)$ is a parameterization of l by arc length, where l is the line in Δ_P with ideal endpoints $(\pm 1, 0)$.

(c) Show that $\gamma(t) = (\tanh(t), \operatorname{sech}(t))$ is a parameterization of l by arc length, where l is the line in the upper half plane model H represented by the upper half of the unit circle. (Hint: Let $(\sqrt{1-y^2}, y)$ be a point on the upper half of the unit circle whose distance from $(0,1)$ is t. Use the formula from problem (2) to solve for y in terms of t.)