Math 104 Homework 6

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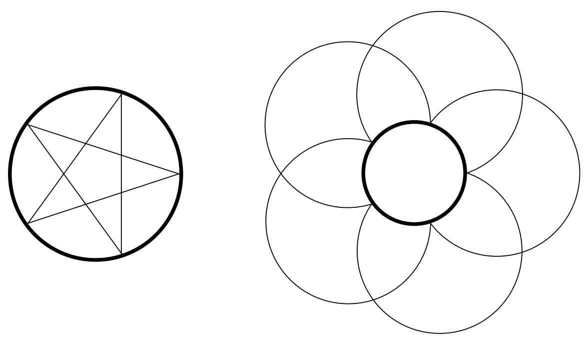
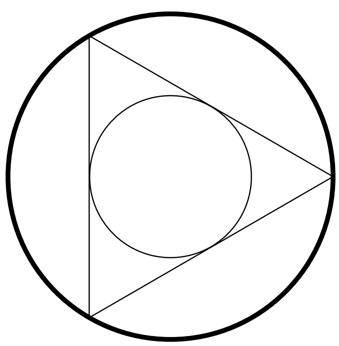
Due Tuesday, March 21, 2006

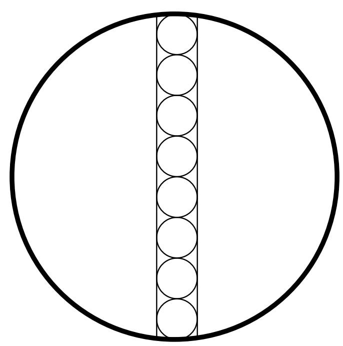
(1) **Pictures of inversions.** In each part of this problem, you are presented with a figure containing a distinguished circle γ (the one drawn with a thicker line). Draw the result of applying the inversion in γ to the entire figure.

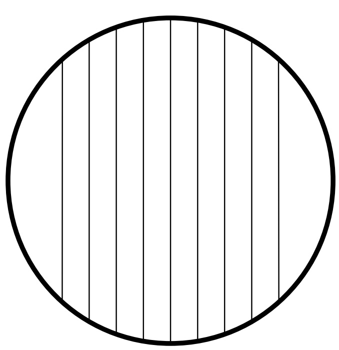
For example, the following images show a pentagram and its inversion in the circumscribed circle:

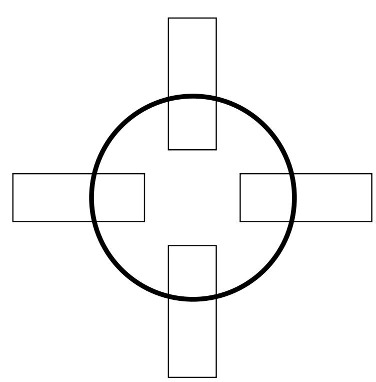


Your drawings do not need to be "perfect" (i.e. you are not required to use a compass, nor to compute centers and radii analytically), but should accurately reflect the properties of inversions applied to lines, circles, angles, etc..

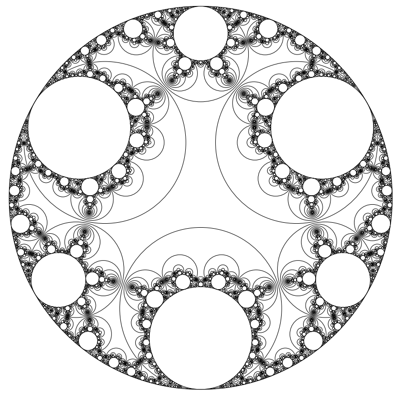
(a)


(b)


(c)


(d)


Extra credit: Draw the inversion of the following figure (which is a union of many circles) in the outermost circle. For this extra credit exercise, your figure will be evaluated based on a high standard of accuracy.



(2) **Asymptotically parallel lines.** Let $P_+ = (0,1)$, $P_- = (0,-1)$, and $Q = (1,0)$. Let l_+ and l_- denote the lines in Δ_K corresponding to the chords P_+Q and P_-Q, respectively. These lines are asymptotically parallel because they have a common ideal endpoint. In this problem you will use a calculation to justify this terminology.

Show that l_+ and l_- are *asymptotic* in the sense that for any $\epsilon > 0$, there are points $X_+ \in l_+$ and $X_- \in l_-$ such that $d_K(X_+, X_-) < \epsilon$.

(Remember: In your solution you must assume you are *given* a small positive real number ϵ, and then find the corresponding pair of points $X_\pm$. Your justification should include a calculation of $d_K(X_+, X_-)$, and an argument showing it is less than ϵ.)

(3) **Ideal triangle calculations.** An *ideal triangle* in the hyperbolic plane is the figure formed by three lines, any two of which are asymptotically parallel. An ideal triangle is *not* a triangle, because it does not have any vertices. In the Klein model, an ideal triangle is a Euclidean triangle inscribed in the unit circle. Let T denote the ideal triangle in Δ_K corresponding to an equilateral Euclidean triangle inscribed in S^1 with $(1,0)$ as one of its vertices.

(a) Find the radius of the hyperbolic circle inscribed in T, i.e. the hyperbolic circle with center $(0,0)$ that is tangent to each of the three edges of T. This radius is the *inradius* of T. (In your solution, include a picture of T and its inscribed circle.)

(b) Let t_1, t_2, t_3 be the three points of tangency of T and its inscribed circle. Then t_1, t_2, t_3 are the vertices of an equilateral triangle in Δ_K. Find the side length of this triangle.

(4) **Coaxial pencils.** Complete exercise **P-13** and part **(a)** of exercise **P-14** from p. 284 of Greenberg. (Note: You can use the facts about radical axes from exercise P-7, including those which were not part of Homework 5. Also, when reading P-7, notice that Greenberg uses the notation $\overline{XT}$ to mean the distance between X and T, which is a real number. In lecture we have used the notation $|XT|$ for this quantity.)

(5) **Pythagorean counterexample.** Show that the pythagorean theorem does *not* hold in hyperbolic geometry by finding an explicit counterexample. (In other words, calculate the side lengths a, b, c of a right triangle and show that they do not satisfy $a^2 + b^2 = c^2$.)