Math 104 Homework 5

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Due Tuesday, March 14, 2006

(1) **Klein-Poincaré isomorphism in coordinates.** This problem will guide you through a proof that the purported isomorphism from the Klein model to the Poincaré model (as described on p. 258 and in Figure 7.33) actually sends chords of the unit circle to orthogonal circular arcs. (Your book uses a different approach to establish the same fact on pp. 258-260.)

(a) Before starting any calculations, should the isomorphism from Δ_K to Δ_P push points *closer* to the origin, *farther* from the origin, or neither? In other words, if (r, θ) are the polar coordinates of a point in Δ_K, and (R, Θ) is the corresponding point in Δ_P, do you expect that $r > R$, $r < R$, or that it depends on the point? Why?

(b) Suppose (r, θ) are the polar coordinates of a point in the Klein model Δ_K. Use similar triangles or any other techniques from Euclidean geometry to show that the corresponding point in the Poincaré model is $(\frac{r}{1+\sqrt{1-r^2}}, \theta)$.

(c) Confirm your suspicion from (1a) using the formula from (1b).

(d) Use the formula from (1b) to establish the formula in Cartesian coordinates: $(x, y) \in \Delta_K$ corresponds to

$$(\frac{x}{1 + \sqrt{1 - x^2 - y^2}}, \frac{y}{1 + \sqrt{1 - x^2 - y^2}}) \in \Delta_P.$$

(e) Now consider the chord of the unit circle connecting the points $P = (s, \sqrt{1 - s^2})$ and $Q = (s, -\sqrt{1 - s^2})$, where $0 < s < 1$. This represents a line in Δ_K. We need to find the corresponding line in Δ_P.

Find the the center C and radius R of the circle γ_{PQ} containing P and Q and which is orthogonal to the unit circle. Your formula for C and R should involve only s.

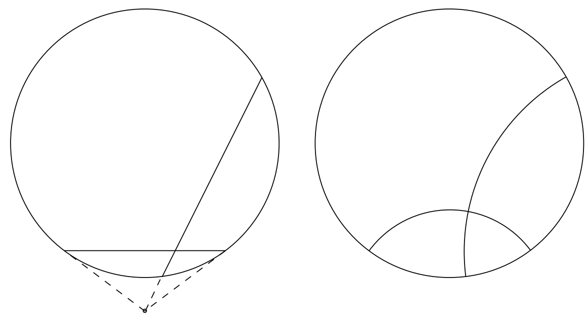
(f) A point on the chord PQ has the form (s, y) where $-\sqrt{1 - s^2} < y < \sqrt{1 - s^2}$. We expect that this point will be mapped to a point on the circle with center C and radius R by the Klein-Poincaré isomorphism.

Confirm this by applying the isomorphism from (1d) to (s, y), and then computing the distance from the resulting point in Δ_P to C. Simplify the expression and show that it agrees with your formula for R from (1e).

(g) Now you have shown that a certain type of chord maps to an orthogonal circular arc. Argue using rotational symmetry that the same must be true for all chords.

(2) **Klein-Poincaré isomorphism in pictures.** Each part of this problem describes a configuration of objects in **one** of the two disk models (Klein, Poincaré). Draw a picture of the corresponding objects in **both** models. The two pictures should be related by the Klein-Poincaré isomorphism.

For example, "a pair of orthogonal lines in the Klein model, neither of which contains the origin" might lead to the two pictures below.



(a) An equilateral triangle inscribed in the unit circle, viewed as three lines in Δ_K.
(b) The set of lines in Δ_K that are represented by chords parallel to the y axis.
(c) The Euclidean circle centered at the origin of radius $\frac{1}{2}$, viewed as a subset of Δ_K.
(d) The Euclidean circle centered at the origin of radius $\frac{1}{2}$, viewed as a subset of Δ_P.
(e) The set of all lines in the Klein model represented by chords with $(0,1)$ as one endpoint.

(3) **Distance in the Klein model.** Exercise K-10 on p. 273 of Greenberg.

(4) **Radical axis.** This problem is about a line determined by a pair of circles in $\mathbb{R}^2$, which has applications to the Poincaré model of the hyperbolic plane.

Let α be a circle with center A and radius r, and let β be a circle with center B and radius s. Assume $A \neq B$, and let C be the unique point on $\overline{AB}$ such that $|AC|^2 - |BC|^2 = r^2 - s^2$. The line through C perpendicular to $\overline{AB}$ is called the *radical axis* of α and β.

(a) It is not clear that this definition of C makes sense; why is there any point with this property, and why is there only one? Suppose $A = (0,0)$ and $B = (t,0)$. Show that $C = (c,0)$ is well-defined by solving the defining equation to determine c.
(b) Show that P lies the radical axis if and only if $|PA|^2 - |PB|^2 = r^2 - s^2$.
(c) Suppose that α and β intersect in two points P and Q. Show that $\overline{PQ}$ is the radical axis.

(Note: this problem is adapted from P-7 on p. 280 of Greenberg.)