Math 104 Homework 3

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Due Thursday, February 23, 2006

(1) **Betweenness and Order.** For real numbers $x, y, z \in \mathbb{R}$, one usually says that y is between x and z if $x < y < z$ or $z < y < x$. Thus our concept of betweenness for real numbers is related to the notion of order.

This problem explores the extent to which a betweenness relation satisfying Hilbert's axioms is like the ordering of the real numbers. *Note: This problem is about betweenness. You may use any of the propositions from the textbook before p. 82, but you should not use congruence in any way.*

Given an ordered pair of distinct points (A, B), let S denote the set of points C such that $A \star C \star B$. Thus S is the *interior* of the segment AB. For the rest of the problem, A and B are fixed.

For a pair of distinct points $X, Y \in S$, let us write $X \prec Y$ if $A \star X \star Y$.

(a) Show that $X \prec Y$ if and only if $X \star Y \star B$.

(b) Suppose $X \prec Y$ and $Y \prec Z$. Show that $X \prec Z$. (This means $\prec$ is a *transitive relation.*)

(c) Suppose $X, Y \in S$. Show that exactly one of the following holds:

(i) $X \prec Y$

(ii) $X = Y$

(iii) $Y \prec X$

(This means $\prec$ is a *trichotomous relation.*)

A binary relation that is transitive and trichotomous is called a *total order.* Thus you have shown that $\prec$ makes the interior of the segment AB into a *totally ordered set*, much as $<$ makes $\mathbb{R}$ into a totally ordered set.

Extra credit: Suppose that you are given two intervals, AB and CD, and $AB \subset CD$. Then we have two total orders ($\prec_{AB}$ and $\prec_{CD}$) on the interior of AB. Show that these two orders actually coincide. Do this lead to a unique total order on the line $\overline{AB}$?

(2) **Congruence and order on segments**. Problems from Chapter 2 of Greenberg:

- Exercise 21 on p. 108
- Exercise 22 on p. 108
- Exercise 23 on p. 108

(3) **Equilateral = Equiangular**. Given $\triangle ABC$, suppose $\measuredangle A \simeq \measuredangle B \simeq \measuredangle C$. Show that $\triangle ABC$ is equilateral, i.e. $AB \simeq AC \simeq BC$. (This is exercise 28 on p. 109.)

(4) **Rational Plane.** Complete exercise 34 on p. 109. Note that "the usual interpretations of the undefined geometric terms used in analytic geometry" means that you can use things like the distance between points in the plane.

For example, two segments AB and CD are congruent if and only if $\|A-B\| = \|C-D\|$ where $\|(x,y)\| = \sqrt{x^2+y^2}$.