1. **Circles.** Let δ denote the Euclidean circle with center $(s,0)$ and radius s, where $0 < s < \frac{1}{2}$.

   (a) Compute the center C and radius S of the hyperbolic circle represented by δ in Δ_P.

   Solution: Let $P = (2s, 0)$. Then OP is a diameter of the hyperbolic circle represented by δ, and the hyperbolic radius is

   $$S = \frac{1}{2} d_P(O,P) = \frac{1}{2}\log\frac{1+2s}{1-2s} = \log\sqrt{\frac{1+2s}{1-2s}}.$$

   The center $C = (x, 0)$ is the midpoint of OP, so we need only find x such that $d_P(O,C) = S$, i.e.

   $$\log\frac{1+x}{1-x} = \log\sqrt{\frac{1+2s}{1-2s}}.$$

   Simplifying, we find

   $$x = \frac{\sqrt{1+2s} - \sqrt{1-2s}}{\sqrt{1+2s} + \sqrt{1-2s}} = \frac{2s}{1+\sqrt{1-4s^2}}.$$

   (b) Let γ denote the hyperbolic circle in Δ_P with center $(0,0)$ and the same hyperbolic radius as δ. Then as Euclidean circles, which has larger radius, δ or γ? Why?

   Solution: Since $O = (0,0) \in \delta$, the hyperbolic circle with center O and radius equal to that of δ will pass through C. Hence the associated Euclidean circle has radius x, where $x = \frac{2s}{1+\sqrt{1-4s^2}}$. Since $s > 0$, we have $\sqrt{1-4s^2} < 1$ and $x > s$, i.e. γ has larger Euclidean radius than δ.

2. **Distorted angles.** Recall that in the Klein model of the hyperbolic plane, lines are represented by Euclidean segments, but the hyperbolic angle between two rays is not necessarily equal to the Euclidean angle.

   Let's call an angle in Δ_K *undistorted* if it is equal to the associated Euclidean angle, and *distorted* otherwise.

   (a) Show that every angle based at $(0,0) \in \Delta_K$ is undistorted.

   Solution: The angle between two rays in Δ_K is defined to be the angle between the corresponding rays in Δ_P (using the isomorphism between these models).

   The isomorphism from Δ_K to Δ_P preserves rays based at $(0,0)$, so an angle formed by rays $\overrightarrow{r_1}$ and $\overrightarrow{r_2}$ (which are radii of S^1) corresponds to the angle formed by the *same* pair of rays in Δ_P. Since Δ_P is conformal, the hyperbolic angle is equal to the Euclidean angle, and the original angle in Δ_K is undistorted.

   (b) Show that for every point $P \in \Delta_K$, there is an undistorted angle based at P.

   Solution: If $P = (0,0)$, this follows from (a). Otherwise, let l denote the diameter of S^1 containing P, and m the chord of S^1 perpendicular (in the Euclidean sense) to l at P.

By definition of orthogonality in Δ_K, l and m are perpendicular, so they form an undistorted right angle at P.

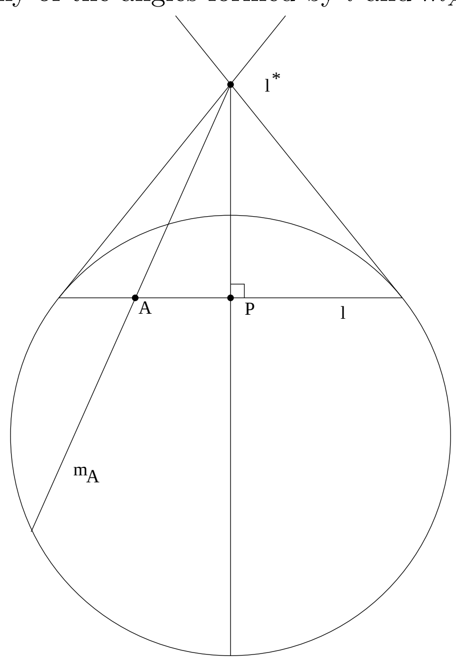
(c) Give an example of a distorted angle in Δ_K (and show that it is distorted).

Solution: Actually, the *only* undistorted angles in Δ_K are the angles based at $(0, 0)$ and the right angles where one ray lies on a diameter. There are several approaches to finding examples of distorted angles.

Here is a construction of an angle that is right in the Klein sense, but not in the Euclidean sense:

Let l be a line in Δ_K that is not a diameter of S^1, and l^* its pole. Let P be the foot of the Euclidean perpendicular to l through l^*, and choose a point A on l distinct from P.

Let m_A be the Klein perpendicular to l at A. Thus the Euclidean extension of m_A passes through through l^*, as in the figure below. However, since the Euclidean perpendicular to l through l^* is unique, and $A \neq P$, m_A is not perpendicular to l in the Euclidean sense. Thus any of the angles formed by l and m_A is distorted.



Here is another approach to this problem, which yields an angle that is right in the Euclidean sense but not in the Klein sense:

Let l and m denote the Euclidean lines defined by the equations $x = \frac{1}{2}$ and $y = \frac{1}{2}$, respectively. Note that l and m are perpendicular in the Euclidean sense. However, the chords of S^1 on l and m define lines in the Klein model that are not perpendicular, because the pole of l lies on the x axis, and therefore not on m. Thus the angles formed by l and m at $(\frac{1}{2}, \frac{1}{2})$ are distorted.

3. **Ultra-ideal points.** Recall that the pole of a line in Δ_K is an ultra-ideal point, and so a set of lines determines a set of ultra-ideal points. Each part of this problem asks you to draw a picture of the set of ultra-ideal points with a certain property (i.e shade a certain subset of the

exterior of the unit circle).

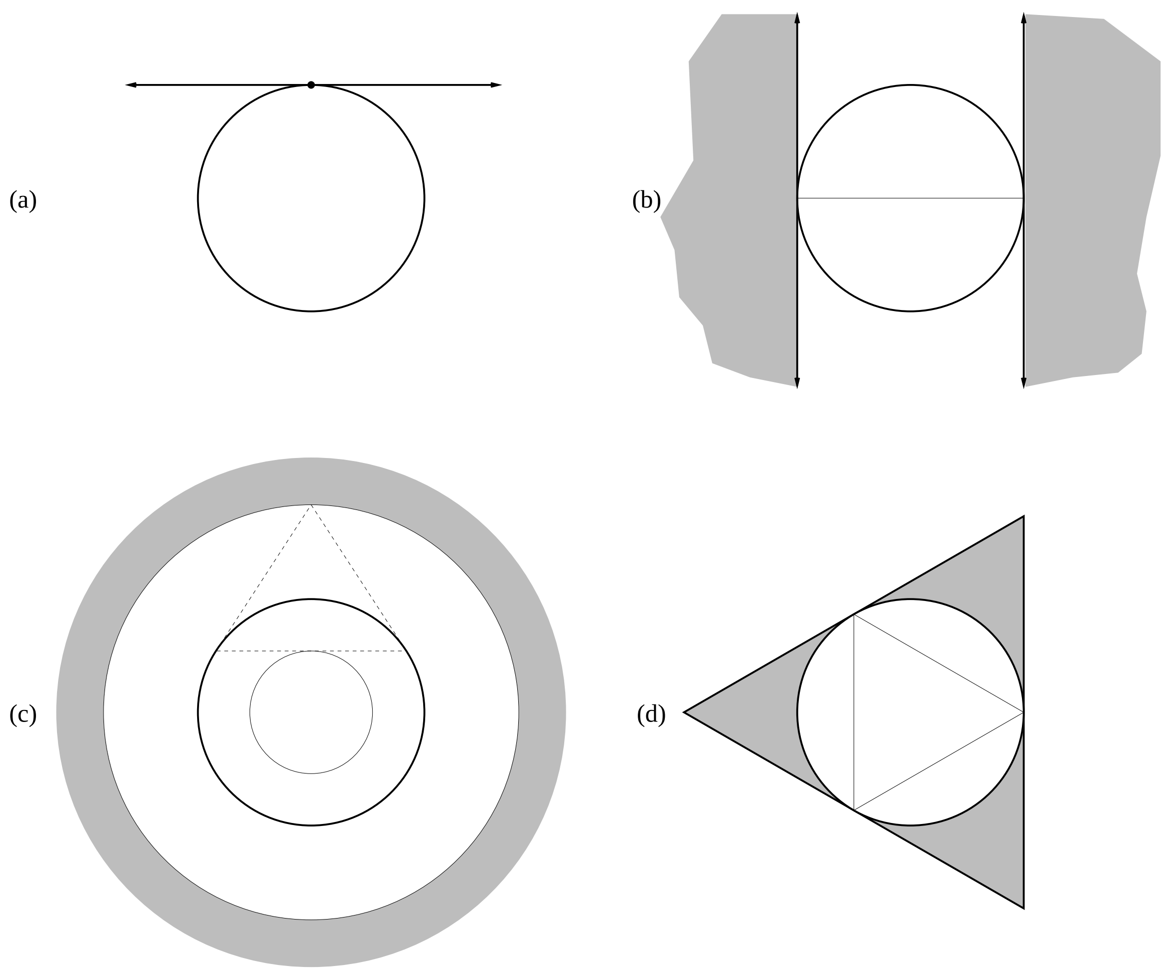
(a) Draw a picture of the set of all ultra-ideal points that are poles of lines having $(0,1)$ as an ideal endpoint.

(b) Let l be the line in Δ_K with ideal endpoints $(-1,0)$ and $(1,0)$. Draw a picture of the set of all ultra-ideal points that are poles of lines intersecting l.

(c) Let $O=(0,0)\in\Delta_K$ and let γ be a hyperbolic circle in Δ_K with center O. (The radius of γ is not important, but for concreteness you can take it to be $\log\sqrt{3}$ if you want.) Draw a picture of the set of all ultra-ideal points that are poles of lines that intersect γ.

(d) Let T be the ideal triangle in Δ_K represented by a Euclidean equilateral triangle inscribed in S^1 with $(1,0)$ as one of its vertices. Draw a picture of the set of all ultra-ideal points that are poles of lines that *do not* intersect T.

Solution:



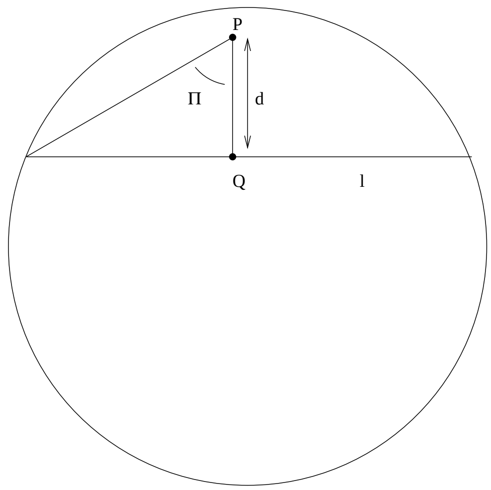
4. **Defect.**

(a) Define the *defect* of a triangle.

Solution: The defect of $\triangle ABC$ is $\delta = \pi - (\measuredangle A + \measuredangle B + \measuredangle C)$.

(b) Define the *angle of parallelism function* $\Pi(d)$, and describe its limiting behavior, i.e. $\Pi(0)$ and $\lim_{d\to\infty} \Pi(d)$. (You do not need to write down a formula for $\Pi(d)$.)

Solution: Let l be a line and P a point whose perpendicular distance from l is d. Let Q be the foot of the perpendicular of l through P, and let $\overrightarrow{r}$ be a limiting parallel ray of l based at P, as in the figure below. Then $\Pi(d)$ is the measure of the angle formed by $\overrightarrow{r}$ and $\overrightarrow{PQ}$.

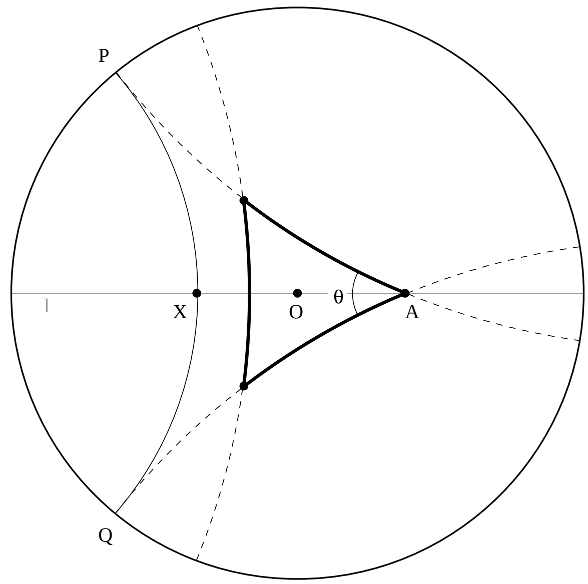


It can be shown (e.g. using the formula $e^{-d} = \tan(\Pi(d)/2)$) that Π is decreasing, $\Pi(0) = \frac{\pi}{2}$,$\lim_{d\to\infty} \Pi(d) = 0$.

(c) Let T_s denote the triangle in Δ_P whose vertices have polar coordinates $(s, 0)$, $(s, \frac{2\pi}{3})$, and $(s, \frac{4\pi}{3})$, where $0 < s < 1$. Thus T_s is an equilateral hyperbolic triangle with center $(0, 0)$. Show that the defect of T_s approaches π as $s \to 1$.

Solution: The three angles of T_s are congruent (because they are related by rotation of Δ_P centered at O), so let θ denote the measure of any one of them. The defect of T_s is $\pi - 3\theta$, so we need only show that $\theta \to 0$ as $s \to 1$.

Let l denote the line in Δ_P represented by the x axis, and let $A = (s, 0)$ be the vertex of T_s on l. Let P and Q denote the ideal endpoints of the rays based at A containing sides of T_s, so P and Q are symmetric with respect to l. Then l_{PQ} meets l orthogonally at a point X, and $\theta = 2\Pi(d_P(X, A))$. (See the figure below.)



Since $d_P(X, A) > d_P(O, A) = \log \frac{1+s}{1-s}$, we have $d_P(X, A) \to \infty$ as $s \to 1$. Thus $\theta = 2\Pi(d_P(X, A)) \to 0$ as $s \to 1$.