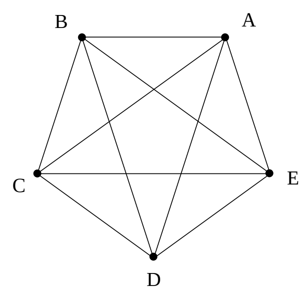
1. Construct a finite model of incidence geometry that has the following strong form of the hyperbolic parallel property: Given any line l and a point P not on l, there exist at least two distinct lines containing P and parallel to l.

Solution: Define the *complete n-point incidence geometry* G_n as follows: $POINTS$ is a set with n elements, and $LINES$ is the set of all pairs of distinct points. In this case the incidence relation is set membership, i.e. point P is incident with line l (which is a set of two points) if and only if $P \in l$.

Then G_n has the strong hyperbolic parallel property as long as $n \geq 5$. The particular cases $n = 3, 4, 5$ were discussed in lecture. A picture of the geometry G_5 where $POINTS = \{A, B, C, D, E\}$ is included below.



2. Suppose we try to construct a new betweenness relation on $\mathbb{R}^2$ as follows: We say C is *between* A and B if and only if C is the midpoint of AB in the usual sense, i.e. if $A = (x, y)$ and $B = (x', y')$, then $A \star C \star B$ iff $C = (\frac{x+x'}{2}, \frac{y+y'}{2})$.

Which betweenness axioms are satisfied by this relation, and which are not? For each axiom that is not satisfied, give a counterexample.

Solution: Let's call the proposed betweenness relation 'midpoint betweenness'.

B1. If $A \star B \star C$, then A, B, and C are distinct points lying on the same line, and $C \star B \star A$.

This axiom is **satisfied** by midpoint betweenness, except possibly the "distinct" requirement. It would be acceptable to say "the definition of midpoint betweenness does not require that A and B are distinct, thus $A = B = C$ is a counterexample".

B2. Given any two distinct points B and D, there exist points A, C, and E lying on $\overline{BD}$ such that $A \star B \star D$, $B \star C \star D$, and $B \star D \star E$.

This axiom is **satisfied** by midpoint betweenness, and in fact there are unique choices for the points A, C, E:

$$C = \frac{1}{2}(B + D)$$
$$A = 2B - D$$
$$E = 2D - B$$

B3. If A, B, and C are distinct points lying on a line, then one and only one of the points is between the other two.

This axiom is **not satisfied** by midpoint betweenness; more specifically, at most one of the points is between the other two, but often none of the three is between the other two. For example, if $A = (0,0)$, $B = (2,0)$, and $C = (6,0)$, then A, B, and C lie on the x-axis, but none is between the other two because the midpoints of AB, BC, and AC are $(1,0)$, $(4,0)$, and $(3,0)$, respectively.

B4. For every line l and for any three points A, B, and C *not* lying on l:

(a) If A and B are on the same side of l and B and C are on the same side of l, then A and C are on the same side of l.

(b) If A and B are on opposite sides of l and B and C are on opposite sides of l, then A and C are on the same side of l.

(c) If A and B are on the same side of l and B and C are on opposite sides of l, then A and C are on opposite sides of l.

This axiom is **not satisfied** by midpoint betweenness. Note that for midpoint betweenness, "A and B lie on the same side of l" means exactly that "$\frac{1}{2}(A+B)$ does not lie on l".

To construct a counterexample, let l be the x-axis, $A = (0,4)$, $B = (0,2)$, and $C = (0,-4)$. Then:

- A and B lie on the same side of l because $\frac{1}{2}(A+B) = (0,3)$ is not on the x-axis
- B and C lie on the same side of l because $\frac{1}{2}(B+C) = (0,-1)$ is not on the x-axis
- But A and C lie on opposite sides of l because $\frac{1}{2}(A+C) = (0,0)$ lies on the x-axis.

Thus (4a) does not hold. Note that (4c) is also false for midpoint betweenness, but (4b) holds.

3. In a geometry with betweenness and congruence, let A, B, C be three distinct points such that $AB \simeq BC \simeq AC$. Prove that A, B, and C are *not* collinear.

Solution: Suppose on the contrary that A, B, and C are collinear. Then by axiom B3, one of them is between the other two. The labels A, B, and C can be permuted without changing the hypotheses, so we may as well assume that $A \star B \star C$.

Then B and C both lie on the ray $\overrightarrow{AB}$, and $AB \simeq AC$. Therefore, by axiom C1, $B = C$, which contradicts the hypothesis that A, B, and C are distinct.

4. (a) Define what is means for a point D to be in the *interior* of a triangle $\triangle ABC$.

Solution: We say D is in the *interior* of triangle $\triangle ABC$ if the following three conditions are satisfied:

1. D and A are on the same side of $\overline{BC}$
2. D and B are on the same side of $\overline{AC}$
3. D and C are on the same side of $\overline{AB}$

(b) A set of points S is called *convex* if for any two distinct points D and E in S, the entire segment DE lies in S. Prove that the interior of a triangle is convex.

Solution: There are many ways to approach the problem. We choose an approach based on the concept of a half plane.

Recall that a *half plane* is the set of all points on one side of a line. We proceed in three steps:

1. *The interior of triangle $\triangle ABC$ is the intersection of three half planes.*

   Let H_A denote the set of all points on the same side of $\overline{BC}$ as A. Let H_B denote the set of all points on the same side of $\overline{AC}$ as B. Let H_C denote the set of all points on the same side of $\overline{AB}$ as C. Then H_A, H_B, and H_C are half planes.

   By definition, D is in the interior of $\triangle ABC$ if and only if it is in each of H_A, H_B, and H_C. Thus the interior of $\triangle ABC$ is the intersection $H_A \cap H_B \cap H_C$.

2. *A half plane is convex.*

   Suppose D and E are in the half plane H and $D \star F \star E$. Since D and E are on the same side of the line l defining H, the segment DE does not meet l. Since $D \star F \star E$, we have $DF \subset DE$, so DF does not meet l. Hence F is on the same side of l as D, i.e. $F \in H$.

3. *If S and T are convex, then $S \cap T$ is convex.*

   Suppose D and E are in $S \cap T$. Then since S is convex, DE is contained in S. Since T is convex, DE is contained in T. Therefore DE is contained in $S \cap T$, and $S \cap T$ is convex.

Now the statement follows easily: By (1), the interior of $\triangle ABC$ is $H_A \cap H_B \cap H_C$, where each of these half planes is convex by (2), so applying (3) twice we find that the interior of $\triangle ABC$ is convex.