Math 104 Quiz 1

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Read these instructions carefully.

- Write your name and "Math 104 Quiz 1" on the front of a blue examination book.
- Read each problem carefully before you attempt to solve it.
- Write your solutions to the problems in the examination book.
- Clearly indicate where your solution to each problem begins and ends.
- Make sure your solutions are clear, concise, and legible.
- A diagram without a written explanation is not a valid solution to any problem.
- You may use propositions from class, reading, or the homework in your solutions, but you must clearly indicate what you are using and how:
  - Not acceptable: *This never happens because of something we proved in lecture.*
  - Acceptable: *An equiangular triangle is equilateral (homework), therefore...*

Do not turn the page until you are told to do so!

1. Construct a finite model of incidence geometry that has the following strong form of the hyperbolic parallel property: Given any line l and a point P not on l, there exist at least two distinct lines containing P and parallel to l.

2. Suppose we try to construct a new betweenness relation on $\mathbb{R}^2$ as follows: We say C is *between* A and B if and only if C is the midpoint of AB in the usual sense, i.e. if $A = (x, y)$ and $B = (x', y')$, then $A \star C \star B$ iff $C = (\frac{x+x'}{2}, \frac{y+y'}{2})$.

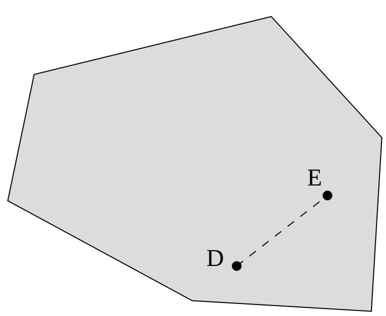
   Which betweenness axioms are satisfied by this relation, and which are not? For each axiom that is not satisfied, give a counterexample.

3. In a geometry with betweenness and congruence, let A, B, C be three distinct points such that $AB \simeq BC \simeq AC$. Prove that A, B, and C are *not* collinear.

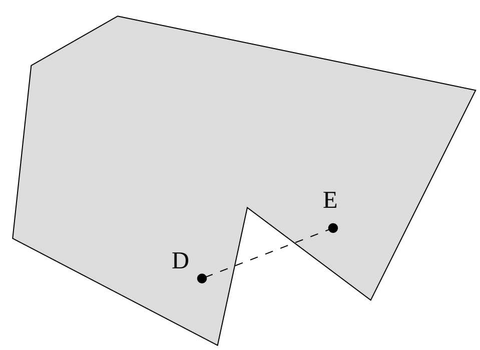
4. (a) Define what is means for a point D to be in the *interior* of a triangle $\triangle ABC$.

   (b) A set of points S is called *convex* if for any two distinct points D and E in S, the entire segment DE lies in S. (See the figure below.)

   Prove that the interior of a triangle is convex.



Convex



Not convex

Axioms

Incidence Geometry

IG1. For every point P and every point Q not equal to P there exists a unique line l incident with P and Q.
IG2. For every line l there exist at least two distinct points that are incident with l.
IG3. There exist three distinct points with the property that no line is incident with all three of them.

Betweenness

B1. If $A \star B \star C$, then A, B, and C are distinct points lying on the same line, and $C \star B \star A$.
B2. Given any two distinct points B and D, there exist points A, C, and E lying on $\overline{BD}$ such that $A \star B \star D$, $B \star C \star D$, and $B \star D \star E$.
B3. If A, B, and C are distinct points lying on a line, then one and only one of the points is between the other two.
B4. For every line l and for any three points A, B, and C *not* lying on l:
  (a) If A and B are on the same side of l and B and C are on the same side of l, then A and C are on the same side of l.
  (b) If A and B are on opposite sides of l and B and C are on opposite sides of l, then A and C are on the same side of l.
  (c) If A and B are on the same side of l and B and C are on opposite sides of l, then A and C are on opposite sides of l. (This actually follows from (4a) and (4b).)

Congruence

C1. If A and B are distinct points and if A' is any point and r is a ray based at A', then there is a unique point B' on r such that $B' \neq A'$ and $AB \simeq A'B'$.
C2. Congruence of segments is reflexive, symmetric, and transitive. That is:
  (a) Every segment is congruent to itself
  (b) If $AB \simeq CD$ then $CD \simeq AB$
  (c) If $AB \simeq CD$ and $CD \simeq EF$ then $AB \simeq EF$
C3. Given any angle $\measuredangle BAC$ and any ray $\overrightarrow{A'B'}$ based at a point A', there is a unique ray $\overrightarrow{A'C'}$ based at A' on a given side of $\overline{A'B'}$ such that $\measuredangle B'A'C' \simeq \measuredangle BAC$.
C4. Congruence of angles is reflexive, symmetric, and transitive. That is:
  (a) Every angle is congruent to itself
  (b) If $\measuredangle A \simeq \measuredangle B$ then $\measuredangle B \simeq \measuredangle A$
  (c) If $\measuredangle A \simeq \measuredangle B$ and $\measuredangle B \simeq \measuredangle C$ then $\measuredangle A \simeq \measuredangle C$
C5. If two sides and the included angle of one triangle are congruent to two sides and the included angle of another triangle (respectively), then the two triangles are congruent. (This is the *SAS congruence criterion.*)

Parallel Properties

- Euclidean Parallel Property: Given a line l and point P not on l, there exists a unique line m containing P such that m is parallel to l.
- Elliptic Parallel Property: Any two lines intersect.
- Hyperbolic Parallel Property: There exists a line l and a point P not on l such that at least two distinct lines contain P and are parallel to l.