Math 52 Final Exam Topic List

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What will be covered

The final exam is cumulative, covering all of the material from Math 52, which is roughly the the first 6 chapters of Strang's book (except section 6.7) and the first two sections of chapter 7.

Material covered since the second exam will be emphasized on the final, but older material will still account for at least half of the questions.

What to expect

The problems on the exam will closely resemble the problems from the homework. The distribution of different types of questions (e.g. calculations vs. theoretical exercises) will also mirror what you have seen on the homework assignments.

A typical question will ask you to apply the methods we have learned to a particular example (perhaps involving some unknown quantities) and draw conclusions about what the results mean. Sometimes a question may involve multiple concepts from different parts of the course, e.g. "Suppose 0 is not an eigenvalue of the square matrix A; why does $A\mathbf{x} = \mathbf{b}$ always have a solution?"

How to prepare

In preparing for the exam, start by reviewing the reading and especially the "worked problems" at the end of each section. Strang includes a list of "conceptual questions" at the end of the book (starting on p. 546); ask yourself these questions after reviewing the reading, and make sure you can confidently answer them. For practice, try a few problems from the textbook, especially those similar to the problems assigned for homework. Solutions to many exercises can be found starting on p. 502.

Since we have discussed a number of connections between the properties of a matrix A and its eigenvalues and eigenvectors, you may find it helpful to refer to the summary of such connections on p. 362. Keep in mind that this table includes some topics we have not discussed, like the singular value decomposition.

Outline of topics covered since Exam 2

(1) Eigenvalues and eigenvectors (§6.1)
- (a) Eigenvalues and eigenvectors are only defined for *square* matrices, because $\mathbf{x}$ and $A\mathbf{x}$ must be vectors of the same size.
- (b) We say $\mathbf{x}$ is an *eigenvector* of an $n \times n$ matrix A with *eigenvalue* λ if

$$A\mathbf{x} = \lambda\mathbf{x}$$

and $\mathbf{x} \neq \mathbf{0}$.

(c) Thus $\mathbf{x}$ is an eigenvector if it and $A\mathbf{x}$ have the *same direction*, i.e. one is a multiple of the other.

(d) Both $\mathbf{x}$ and λ are "unknowns" in the equation $A\mathbf{x} = \lambda\mathbf{x}$; it is not a linear system like $A\mathbf{x} = \mathbf{b}$.

(e) $A\mathbf{x} = \lambda\mathbf{x}$ is equivalent to $\mathbf{x} \in N(A - \lambda I)$, so λ is an eigenvector of A when $(A - \lambda I)$ is *singular*.

(f) 0 is an eigenvalue of A if and only if A is singular; eigenvectors with eigenvalue 0 are vectors in $N(A)$.

(g) The roots of the *characteristic equation* $\det(A - \lambda I) = 0$ are the eigenvalues of A, because these are the values of λ for which $(A - \lambda I)$ is singular.

(h) $\det(A - \lambda I)$ is a polynomial of degree n in λ (the *characteristic polynomial*), so there are n eigenvalues, counted with multiplicity.

(i) To find linearly independent *eigenvectors* with eigenvalue λ, compute a basis for the null space of $(A - \lambda I)$. (Therefore, once you know the eigen*values*, it is easy to find the associated eigen*vectors*.)

(j) The sum of the n eigenvalues of A is equal to the *trace* of A (the sum of its diagonal entries), i.e.

$$\sum_i \lambda_i = \sum_i a_{ii}.$$

(k) The product of the n eigenvalues of A is equal to the determinant of A i.e.

$$\det(A) = \lambda_1 \, \lambda_2 \, \cdots \, \lambda_n$$

(2) Diagonalizability (§6.2)

(a) We say A is *diagonalizable* if it has n linearly independent eigenvectors. Equivalently, there is a basis of $\mathbb{R}^n$ consisting of eigenvectors of A.

(b) Some matrices are diagonalizable (like $\left(\begin{smallmatrix} 2 & 1 \\ 1 & 2 \end{smallmatrix}\right)$), while others are not (like $\left(\begin{smallmatrix} 2 & 1 \\ 0 & 2 \end{smallmatrix}\right)$).

(c) A "random" matrix is likely to be diagonalizable, and any matrix can be made diagonalizable by changing its entries by an arbitrarily small amount.

(d) If A has n distinct eigenvalues, then A is diagonalizable; this follows from two facts:

(i) Every eigenvalue has at least one associated eigenvector

(ii) Eigenvectors with different eigenvalues are linearly independent

(e) If λ is an eigenvalue of A, its *algebraic multiplicity* is the number of times it appears as a root of $\det(A - \lambda I)$. For example, if $\det(A - \lambda I) = \lambda^2(\lambda + 1)(\lambda - 2)^3$ then the algebraic multiplicities are:

λ	AM
0	2
-1	1
2	3

(f) If λ is an eigenvalue of A, its *geometric multiplicity* is the maximal number of linearly independent eigenvectors with eigenvalue λ, i.e.

$$GM = \dim N(A - \lambda I).$$

For example, here are two 2×2 matrices whose only eigenvalue is 2 (hence $AM = 2$), but where the geometric multiplicities differ.

A	GM of $\lambda = 2$
$\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$	2
$\begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}$	1

(g) A is diagonalizable exactly when $AM = GM$ for each eigenvalue. Since $GM \leq AM$, non-diagonalizable matrices are exactly those with too few eigenvectors.

(3) Diagonalization (§6.2)

(a) Suppose A is a diagonalizable matrix with eigenvalues $\lambda_1, \ldots, \lambda_n$ and associated eigenvectors $\mathbf{x}_1, \ldots, \mathbf{x}_n$, i.e.

$$A\mathbf{x}_i = \lambda_i \mathbf{x}_i.$$

(b) Let

$$\Lambda = \begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{pmatrix}.$$

This is the diagonal *eigenvalue matrix*.

(c) Let

$$S = \begin{pmatrix} \mathbf{x}_1 & \mathbf{x}_2 & \cdots & \mathbf{x}_n \end{pmatrix}$$

This is the *eigenvector matrix*, which is invertible because its columns form a basis of $\mathbb{R}^n$.

(d) Since $A\mathbf{x}_i = \lambda_i \mathbf{x}_i$, we have

$$A = S\Lambda S^{-1}.$$

(e) Conversely, if $A = S\Lambda S^{-1}$, then the columns of S are eigenvectors and the diagonal entries of Λ are the eigenvalues.

(f) Since $A^k = S\Lambda^k S^{-1}$, the matrix A^k has the same eigenvectors as A, with eigenvalues $\lambda_1^k, \ldots, \lambda_n^k$.

(g) If 0 is not an eigenvalue, then A is invertible and $A^{-1} = S\Lambda^{-1}S^{-1}$, so A^{-1} has the same eigenvectors as A, with eigenvalues $1/\lambda_i, \ldots, 1/\lambda_n$.

(h) $A^k \to 0$ as $k \to \infty$ if all of the eigenvalues of A satisfy $|\lambda_i| < 1$. (This is true even if A is not diagonablizable, but for diagonalizable matrices it follows from $A = S\Lambda S^{-1}$.)

(4) Application: Difference Equations (§6.2)

(a) Define a sequence of vectors starting with $\mathbf{u}_0$ and applying the rule

$$\mathbf{u}_{k+1} = A\mathbf{u}_k.$$

(b) If $\mathbf{x}_i$ is an eigenvector of A with eigenvalue λ_i, then

$$\mathbf{u}_k = \lambda_i^k \mathbf{x}_i$$

is a solution of $\mathbf{u}_{k+1} = A\mathbf{u}_k$.

(c) If $|\lambda_i| < 1$, then the corresponding solution decays as $k \to \infty$, while if $|\lambda_i| > 1$ the solution blows up.

(d) If $\lambda_i = 1$, then any multiple of $\mathbf{x}_i$ is a steady-state solution, because $A\mathbf{x}_i = \mathbf{x}_i$.

(e) If A is diagonalizable, then its eigenvectors form a basis of $\mathbb{R}^n$, so any starting vector $\mathbf{u}_0$ is a linear combination of them:

$$\mathbf{u}_0 = c_1\mathbf{x}_1 + \ldots c_n\mathbf{x}_n$$

The corresponding solution to $\mathbf{u}_{k+1} = A\mathbf{u}_k$ is therefore

$$\mathbf{u}_k = c_1\lambda_1^k\mathbf{x}_1 + \ldots c_n\lambda_n^k\mathbf{x}_n$$

(f) The matrix form of this solution is

$$\mathbf{u}_k = A^k\mathbf{u}_0 = S\Lambda^k S^{-1}\mathbf{u}_k = S\Lambda^k\mathbf{c}.$$

(g) This method allows us to find an explicit formula for the Fibonacci numbers (0, 1, 1 , 2, 3, 5, 8, 13, 21, 33, ...), which obey the second-order relation

$$F_{k+2} = F_{k+1} + F_k.$$

This can be made into a first-order difference equation by considering the vector

$$\mathbf{u}_k = \begin{pmatrix} F_{k+1} \\ F_k \end{pmatrix}$$

which satisfies

$$\mathbf{u}_{k+1} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \mathbf{u}_k.$$

(h) The k^{th} Fibonacci number is approximately

$$F_k \approx \frac{1}{\sqrt{5}} \left(\frac{1+\sqrt{5}}{2} \right)^k$$

because $\frac{1+\sqrt{5}}{2}$ is the only eigenvalue of $\left(\begin{smallmatrix} 1 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ larger than 1. (The factor $\frac{1}{\sqrt{5}}$ comes from the initial conditions $F_0 = 0$, $F_1 = 1$.)

(5) Application: Differential Equations (§6.3)

(a) Given an $n \times n$ matrix A, consider the system of differential equations

$$\frac{d\mathbf{u}}{dt} = A\mathbf{u}$$

where

$$\mathbf{u}(t) = \begin{pmatrix} u_1(t) \\ \vdots \\ u_n(t) \end{pmatrix}.$$

This system is linear, first-order, and has constant coefficients.

(b) If $\mathbf{x}_i$ is an eigenvector of A with eigenvalue λ_i, then

$$\mathbf{u}(t) = e^{\lambda_i t}\mathbf{x}_i$$

is a solution of $\frac{d\mathbf{u}}{dt} = A\mathbf{u}$. It is a *pure exponential solution*, where $\mathbf{u}(t)$ moves along the line of multiples of $\mathbf{x}_i$.

(c) If $\text{Re}(\lambda_i) < 0$, then the corresponding solution decays as $t \to \infty$, while if $\text{Re}(\lambda_i) > 0$ the solution blows up.

(d) If $\lambda_i = 0$, then any multiple of $\mathbf{x}_i$ is a steady-state solution, because $A\mathbf{x}_i = 0$.

(e) If A is diagonalizable, then its eigenvectors form a basis of $\mathbb{R}^n$, so any initial condition $\mathbf{u}(0)$ is a linear combination of them:

$$\mathbf{u}(0) = c_1\mathbf{x}_1 + \dots c_n\mathbf{x}_n$$

The corresponding solution to $\frac{d\mathbf{u}}{dt} = A\mathbf{u}$ is therefore

$$\mathbf{u}(t) = c_1 e^{\lambda_1 t}\mathbf{x}_1 + \dots c_n e^{\lambda_n t}\mathbf{x}_n$$

(f) The matrix form of this solution is

$$\mathbf{u}(t) = e^{At}\mathbf{u}(0) = Se^{\Lambda t}S^{-1}\mathbf{u}(0) = Se^{\Lambda t}\mathbf{c}.$$

(g) The matrix exponential e^A is defined by

$$e^A = I + A + \frac{1}{2}A^2 + \dots = \sum_{k=1}^{\infty} \frac{1}{k!}A^k$$

and this sum converges for all $n \times n$ matrices A.

(h) The exponential of a diagonal matrix is

$$e^{\begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{pmatrix}} = \begin{pmatrix} e^{\lambda_1} & & & \\ & e^{\lambda_2} & & \\ & & \ddots & \\ & & & e^{\lambda_n} \end{pmatrix}.$$

(i) Higher-order linear equations with constant coefficients can be reduced to systems of first-order equations; for example, to solve the equation

$$y'' + by' + ky = 0$$

we consider the vector function

$$\mathbf{u}(t) = \begin{pmatrix} y'(t) \\ y(t) \end{pmatrix}$$

which satisfies

$$\frac{d\mathbf{u}}{dt} = \begin{pmatrix} -b & -k \\ 1 & 0 \end{pmatrix}\mathbf{u}.$$

(j) For a 2×2 matrix A, the solutions of $\frac{d\mathbf{u}}{dt} = A\mathbf{u}$ will be stable (decay as $t \to \infty$) if the *trace* of A is negative and the *determinant* of A is positive:

$$\mathrm{tr}(A) < 0 \text{ and } \det(A) > 0$$

(6) Symmetric Matrices and the Spectral Theorem (§6.4)

(a) Every symmetric matrix A is diagonalizable.

(b) The eigenvalues of a symmetric matrix are *real*, and its eigenvectors are *orthogonal*.

(c) If the eigenvectors are chosen to have unit length, they form an orthonormal basis of $\mathbb{R}^n$, so the eigenvector matrix Q is orthogonal. Thus diagonalization becomes $A = Q\Lambda Q^T$ for a symmetric matrix A, which is the *spectral theorem*.

(d) The spectral theorem allows us to write A as a combination of projections,

$$A = \lambda_1 P_1 + \cdots + \lambda_n P_n$$

where P_i is the projection onto the span of the eigenvector $\mathbf{q}_i$.

(e) The pivots and the eigenvalues of a symmetric matrix are different, but are both real. Furthermore, the number of positive (resp. negative) pivots is equal to the number of positive (resp. negative) eigenvalues. The number of zero eigenvalues is the dimension of the null space (which one might be tempted to call "the number of zero pivots").

(7) Positive Definite Matrices (§6.5)

(a) A matrix is *positive definite* if it is symmetric and its eigenvalues are positive, i.e. $\lambda_i > 0$.

(b) A symmetric matrix is positive definite if and only if its pivots are positive. This is the *pivot test*, which works because the pivots and eigenvalues of a symmetric matrix have the same signs.

(c) A symmetric matrix A is positive definite if and only if its upper-left determinants $d_1, \ldots, d_n$ are positive, where

$$\begin{aligned} d_1 &= \det(a_{11}) = a_{11} \\ d_2 &= \det \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \\ &\vdots \\ d_{n-1} &= \det \begin{pmatrix} a_{11} & \cdots & a_{1,(n-1)} \\ \vdots & & \vdots \\ a_{(n-1),1} & \cdots & a_{(n-1),(n-1)} \end{pmatrix} \\ d_n &= \det(A) \end{aligned}$$

This is the *determinant test*, which works because the pivots are ratios of these determinants.

(d) A symmetric matrix A is positive definite if and only if $\mathbf{x}^T A\mathbf{x} > 0$ for all nonzero vectors $\mathbf{x}$. This is the *quadratic form test*.

(e) The quantity $\mathbf{x}^T A\mathbf{x}$ is a *quadratic form*, meaning that it is a quadratic expression in the components of $\mathbf{x}$; specifically,

$$\mathbf{x}^T A\mathbf{x} = \sum_{i,j} a_{ij}x_i x_j \text{ where } \mathbf{x} = \begin{pmatrix} x_1 \\ \vdots \\ \mathbf{x}_n \end{pmatrix}.$$

(f) The *quadratic form test* works because $\mathbf{x}^T A\mathbf{x}$ can be written as a sum of squares where the coefficients are the pivots; this is clearly positive if all of the pivots are positive, and can be zero or negative otherwise.

(g) For a 2×2 symmetric matrix A, we have the LDL^T decomposition

$$\begin{pmatrix} a & b \\ b & c \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \frac{b}{a} & 1 \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & \frac{ac-b^2}{a} \end{pmatrix} \begin{pmatrix} 1 & \frac{b}{a} \\ 0 & 1 \end{pmatrix}$$

so the pivots are a and $\frac{ac-b^2}{a}$. Similarly, the quadratic form $\mathbf{x}^T A\mathbf{x}$ is a sum of squares:

$$\mathbf{x}^T A\mathbf{x} = ax_1^2 + 2bx_1x_2 + cx_2^2 = a\left(x_1 + \frac{b}{a}x_2\right)^2 + \left(\frac{ac-b^2}{a}\right)(x_2)^2$$

(h) If A is positive definite, then the graph of $\mathbf{x}^T A\mathbf{x}$ is a paraboloid. Its minimum is at $\mathbf{x} = \mathbf{0}$.

(i) If A has both positive and negative eigenvalues, then the graph of $\mathbf{x}^T A\mathbf{x}$ is saddle-shaped.

(j) If A is positive definite, the equation $\mathbf{x}^T A\mathbf{x} = 1$ defines an ellipsoid in $\mathbb{R}^n$. The principal axes of the ellipsoid are the lines containing the eigenvectors $\mathbf{q}_1, \ldots, \mathbf{q}_n$ of A, and the half-axis lengths are $1/\sqrt{\lambda_i}$. This comes from $A = Q\Lambda Q^T$.

(k) Thus one can draw the ellipse defined by $ax^2 + 2bxy + cy^2$ by finding the eigenvalues and eigenvectors of $A = \left(\begin{smallmatrix} a & b \\ b & c \end{smallmatrix}\right)$.

(l) Positive definite matrices generalize the second-derivative test from calculus: A critical point of a real-valued function $f(x_1, \ldots, x_n)$ is a minimum if the Hessian matrix

$$H_{ij} = \frac{\partial^2 f}{\partial x_i \partial x_j}$$

is positive definite. Because partial derivatives commute, the Hessian is always symmetric.

(8) Similar Matrices and Jordan's Theorem (§6.6)

(a) Two $n \times n$ matrices A and B are *similar* if $B = M^{-1}AM$ for some invertible matrix M.

(b) Similarity is an equivalence relation, meaning it is

(i) Symmetric: A similar to B iff B similar to A

(ii) Transitive: A similar to B and B similar to C implies A similar to C

(c) A is diagonalizable if and only if it is similar to a diagonal matrix Λ.

(d) Similar matrices have the same:

(i) eigenvalues,
(ii) algebraic and geometric multiplicities,
(iii) trace, and
(iv) determinant.

(e) The eigenvectors of A and of $B = M^{-1}AM$ are different, but related; if $\mathbf{x}$ is an eigenvector of A with eigenvalue λ, then $M^{-1}\mathbf{x}$ is an eigenvalue of B with eigenvalue λ.

(f) If A has *distinct* eigenvalues $\lambda_1, \ldots, \lambda_n$, then its *similarity class* (i.e. the set of all matrices similar to A) consists of all matrices with eigenvalues $\lambda_1, \ldots, \lambda_n$.

(g) If A has a repeated eigenvalue, then another matrix with the same eigenvalues may or may not be similar to A. For example,

$$A = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$$

have only one eigenvalue ($\lambda = 2$) but are not similar. In fact, B is not similar to any matrix except itself.

(h) Even if A and B have the same eigenvalues and the same number of eigenvectors, they may not be similar. For example

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

have zero as their only eigenvalue, each has two independent eigenvectors, but $A^2 = 0$ while $B^2 \neq 0$ so they are not similar.

(i) To classify the different similarity classes of matrices, we find a good representative of each class, generalizing the diagonal matrix Λ similar to any diagonalizable matrix.

(j) The $k \times k$ *Jordan block* $J_k(\lambda)$ has λ as its only eigenvalue, and has one eigenvector:

$$J_k(\lambda) = \begin{pmatrix} \lambda & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & \lambda \end{pmatrix}$$

(k) *Jordan's Theorem* says that every matrix A is similar to a block-diagonal matrix J of Jordan blocks. Furthermore J is unique up to re-ordering the blocks, and is called the *Jordan form* of A. The numbers λ_i in on the diagonals of the blocks are the eigenvalues of A, while the number of blocks is the number of eigenvectors.

(l) A is diagonalizable if and only if its Jordan form has n blocks, each of which must then have size 1×1.

(9) Linear Transformations (§7.1-7.2)

(a) A map $T : V \to W$ (where V and W are vector spaces) is a *linear transformation* if both of these conditions hold:
  (i) $T(c\mathbf{v}) = cT(\mathbf{v})$ for all $c \in \mathbb{R}$ and $\mathbf{v} \in V$
  (ii) $T(\mathbf{v}_1 + \mathbf{v}_2) = T(\mathbf{v}_1) + T(\mathbf{v}_2)$ for all $\mathbf{v}_1, \mathbf{v}_2 \in V$

(b) A linear transformation is also called a *linear map*, or *linear mapping*, or we may simply say that T *is linear*.

(c) A linear map T satisfies $T(\mathbf{0}) = \mathbf{0}$

(d) Examples of linear maps include:
  (i) $T : \mathbb{R}^n \to \mathbb{R}^m$ defined by $T(\mathbf{x}) = A\mathbf{x}$ where A is an $m \times n$ matrix
  (ii) The identity map $Id : V \to V$, defined by $Id(\mathbf{v}) = \mathbf{v}$ for all $\mathbf{v}$
  (iii) The zero map $Z : V \to W$, defined by $Z(\mathbf{v}) = \mathbf{0}$ for all $\mathbf{v}$
  (iv) A rotation $R : \mathbb{R}^2 \to \mathbb{R}^2$ of the plane about the origin
  (v) Projection $P : \mathbb{R}^2 \to \mathbb{R}^2$ of the plane onto a line through the origin

(e) The following maps are not linear:
  (i) The determinant map, $\det : M_{n\times n} \to \mathbb{R}$
  (ii) The length map $L : \mathbb{R}^n \to \mathbb{R}$, where $L(\mathbf{v}) = \|\mathbf{v}\|$
  (iii) A shift map $T : V \to V$, where $T(\mathbf{v}) = \mathbf{v} + \mathbf{v}_0$ and $\mathbf{v}_0 \neq 0$

(f) If $\mathbf{v}_1, \dots, \mathbf{v}_n$ is a basis of V, then any vector $\mathbf{v} \in V$ can be expressed as

$$\mathbf{v} = c_1\mathbf{v}_1 + \dots c_n\mathbf{v}_n$$

for unique real numbers $c_1, \dots, c_n$. These numbers are the *coordinates* of $\mathbf{v}$ with respect to the basis $\mathbf{v}_1, \dots, \mathbf{v}_n$.

(g) If $\mathbf{v}_1, \dots, \mathbf{v}_n$ is a basis of V, then any linear map $T : V \to W$ is uniquely determined by the vectors $T(\mathbf{v}_1), \dots, T(\mathbf{v}_n) \in W$

(h) Choosing an *input basis* $\mathbf{v}_1, \dots, \mathbf{v}_n$ of V and an *output basis* $\mathbf{w}_1, \dots, \mathbf{w}_m$ of W allows us to associate a matrix A to a linear transformation T; the entries of A are the coordinates of $T(\mathbf{v}_i)$:

$$\begin{aligned}
T(\mathbf{v}_1) &= a_{11}\mathbf{w}_1 + a_{21}\mathbf{w}_2 + \cdots + a_{m1}\mathbf{w}_m \\
T(\mathbf{v}_2) &= a_{12}\mathbf{w}_1 + a_{22}\mathbf{w}_2 + \cdots + a_{m2}\mathbf{w}_m \\
&\vdots \\
T(\mathbf{v}_n) &= a_{1n}\mathbf{w}_1 + a_{2n}\mathbf{w}_2 + \cdots + a_{mn}\mathbf{w}_m
\end{aligned}$$

Thus A is an $m \times n$ matrix whose j^{th} column gives the coordinates of $T(\mathbf{v}_j)$ with respect to $\mathbf{w}_1, \dots, \mathbf{w}_m$.

$$A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix}$$

(i) If $T : V \to V$ is a linear transformation from a vector space to itself, and $\mathbf{v}_1, \dots, \mathbf{v}_n$ is a basis of eigenvectors, then the matrix associated to T (using $\mathbf{v}_1, \dots, \mathbf{v}_n$ as both the input and output basis vectors) is diagonal.