Math 52 Final Exam
Fall 2005
David Dumas

Name (print legibly!): ______________________________

Circle your section: Section 1 (MWF 2pm) Section 2 (TTh 1pm)

Answer the questions in the spaces provided on the question sheets. If you run out of room, continue on the back of the page.
Show your work and circle your final answer.

The exam has **8** questions and **14** pages, including this cover page.

Question	Points	Score
1	18	
2	15	
3	20	
4	15	
5	15	
6	15	
7	10	
8	16	
Total:	124	

Do not open the exam until you are told to do so!

1. This problem concerns the matrix

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 2 & 0 & 2 \\ 1 & 3 & 0 \end{pmatrix}.$$

(a) (5 points) Compute the rank of A and the dimensions of the four subspaces. Put your answers in this table:

$\operatorname{rank}(A)$	$\dim C(A)$	$\dim N(A)$	$\dim C(A^T)$	$\dim N(A^T)$

(b) (5 points) Circle the statements that are true:

i. A has full row rank

ii. $A\mathbf{x} = \mathbf{b}$ has at least one solution for any $\mathbf{b}$

iii. For some $\mathbf{b}$, $A\mathbf{x} = \mathbf{b}$ has infinitely many solutions

iv. Changing the last row of A might change the rank of A

v. Changing the last column of A might change the rank of A

This page is a continuation of problem 1. $A = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 2 & 0 & 2 \\ 1 & 3 & 0 \end{pmatrix}$

Parts (c) and (d) are about the linear system $A\mathbf{x} = \mathbf{b}$ where $\mathbf{b} = \begin{pmatrix} -3 \\ -2 \\ 4 \\ -7 \end{pmatrix}$.

(c) (3 points) Does this system have a solution? *Circle your answer.* i. YES ii. NO

(d) (5 points) If you answered YES to part (c), find the **general solution**. If you answered NO, find a **least squares solution**, i.e. a vector $\hat{\mathbf{x}}$ that makes $\|A\hat{\mathbf{x}} - \mathbf{b}\|$ as small as possible.

2. Decide whether each of the following matrices is positive definite or not. Clearly indicate what criteria you use in each case.

   (a) (3 points) $\begin{pmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix}$ Positive definite? i. YES ii. NO

   (b) (3 points) $\begin{pmatrix} 2 & 0 & x \\ 0 & 2 & 0 \\ x & 0 & 2 \end{pmatrix}$ Positive definite? i. YES ii. NO iii. DEPENDS on x

   (c) (3 points) $\begin{pmatrix} 1 & 1 & 1 \\ 1 & 3 & 1 \\ 1 & 1 & 1 \end{pmatrix}$ Positive definite? i. YES ii. NO

   (d) (3 points) $\begin{pmatrix} 3 & 0 & -1 \\ 0 & 2 & 0 \\ -1 & 0 & 3 \end{pmatrix}$ Positive definite? i. YES ii. NO

   (e) (3 points) $\begin{pmatrix} 5 & 4 & 0 & 0 \\ 4 & 3 & 0 & 0 \\ 0 & 0 & 3 & 4 \\ 0 & 0 & 4 & 5 \end{pmatrix}$ Positive definite? i. YES ii. NO

3. Consider a sequence of real numbers $y_0, y_1, y_2, \ldots$ satisfying the rule

$$y_{k+2} = y_{k+1} + 2\, y_k.$$

Such a sequence might begin $0, 1, 1, 3, 5, 11, 21, 43, 85 \ldots$.

(a) (5 points) Let $\mathbf{u}_k = \begin{pmatrix} y_{k+1} \\ y_k \end{pmatrix}$. Find a 2×2 matrix A such that $\mathbf{u}_{k+1} = A\mathbf{u}_k$.

(b) (5 points) Find the eigenvalues λ_1, λ_2 of the matrix A.

This page is a continuation of problem 3.

(c) (5 points) Are there nonzero starting values y_0 and y_1 for which the sequence y_k decays to zero, i.e. $\lim_{k\to\infty} y_k = 0$? *Circle your answer.* i. YES ii. NO

If you answered YES, give an example of such starting values for y_0 and y_1. If you answered NO, explain why such decay is impossible.

(d) (5 points) Suppose the sequence begins with $y_0 = 1$ and $y_1 = 2$. Find an exact formula for y_k.

4. This question concerns the matrix

$$B = \begin{pmatrix} 1 & 1 & 2 \\ 1 & -1 & 2c \\ 0 & 0 & c \end{pmatrix}$$

(a) (5 points) For what values of c are the columns of B linearly **dependent**?

(b) (5 points) Apply the Gram-Schmidt algorithm to the columns of B to find an orthonormal basis for their span when $c = 3$.

This page is a continuation of problem 4.

(c) (5 points) Find the matrix R such that $B = QR$, where Q is the matrix whose columns are the orthonormal basis vectors $\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3$ from part (b) and $c = 3$.

5. Let P_3 denote the vector space of polynomials in one variable of degree at most 3 (with real coefficients). Consider the two linear transformations

$$Id : P_3 \to P_3 \text{ defined by } Id(f) = f$$
$$T : P_3 \to P_3 \text{ defined by } T(f) = \frac{df}{dx}$$

Thus for example $Id(1 + 3x^3) = 1 + 3x^3$ and $T(1 + 3x^3) = 9x^2$.

(a) (5 points) Find the matrix of T using $\mathbf{v}_0 \ldots \mathbf{v}_3$ as both the input basis and the output basis, where

$$\mathbf{v}_0 = 1 \quad \mathbf{v}_1 = x \quad \mathbf{v}_2 = x^2 \quad \mathbf{v}_3 = x^3$$

(b) (5 points) Find the matrix of T using $\mathbf{w}_0 \ldots \mathbf{w}_3$ as both the input basis and the output basis, where

$$\mathbf{w}_0 = 1 \quad \mathbf{w}_1 = 1 + x \quad \mathbf{w}_2 = (1 + x)^2 = 1 + 2x + x^2 \quad \mathbf{w}_3 = (1 + x)^3 = 1 + 3x + 3x^2 + x^3.$$

This page is a continuation of problem 5.

(c) (5 points) Find the matrix of Id using $\mathbf{w}_0 \ldots \mathbf{w}_3$ as the input basis and $\mathbf{v}_0 \ldots \mathbf{v}_3$ as the output basis.

6. Suppose S is a 3×3 **symmetric** matrix of **rank 2**, and

$$S \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}.$$

(a) (5 points) Give an example of such a matrix that is **not** a projection.

(b) (5 points) Give an example of such a matrix that **is** a projection.

(c) (5 points) Could the first column of S be $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$? *Circle your answer.* i. YES ii. NO

If you answered YES, given an example of such a matrix; if you answered NO, explain why.

7. Let

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 5 \end{pmatrix}.$$

Each of the following matrices has some blank entries. If it is possible to fill in the blanks and create a matrix **similar** to A, then circle YES and give an example of how you might fill in the blank entries. Otherwise circle NO and explain why it is impossible to create a matrix similar to A by filling in the blanks.

(a) (5 points) $B = \begin{pmatrix} 2 & \cdot & \cdot \\ \cdot & 4 & \cdot \\ \cdot & \cdot & 2 \end{pmatrix}$ Might be similar to A? i. YES ii. NO

(b) (5 points) $C = \begin{pmatrix} 3 & \cdot & \cdot \\ \cdot & 3 & \cdot \\ \cdot & \cdot & 3 \end{pmatrix}$ Might be similar to A? i. YES ii. NO

8. Let F denote the subspace of $\mathbb{R}^4$ consisting of vectors $\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix}$ satisfying $x - y = z - w$.

(a) (4 points) Find a basis for $F^{\perp}$.

(b) (4 points) Find a basis for F.

This page is a continuation of problem 8.

(c) (4 points) Find the projection of $\mathbf{b} = \begin{pmatrix} 1 \\ 2 \\ 1 \\ 2 \end{pmatrix}$ onto $F^{\perp}$.

(d) (4 points) Find the projection of $\mathbf{b} = \begin{pmatrix} 1 \\ 2 \\ 1 \\ 2 \end{pmatrix}$ onto F.