1. Using cofactors, the big formula, pivots, or any other (valid) method, compute the following:

   (a) (4 points) $\begin{vmatrix} 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 \end{vmatrix}$

   Solution: Two exchanges (columns 1 and 5, then 2 and 4) turn this into $-I$, so the determinant is $|-I| = (-1)^5 = \boxed{\text{-1}}$.

   (b) (4 points) $\begin{vmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{vmatrix}$

   Solution: Apply the cofactor formula for the first row:

   $$\begin{vmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{vmatrix} - \begin{vmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{vmatrix}$$

   Since there is a repeated row in the second term, it is zero. We can apply the cofactor formula again to the first term (row 1):

   $$\begin{vmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} - \begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} = 0 - 1 = \boxed{\text{-1}}$$

   (This matrix is A_4, the 4×4 tri-diagonal matrix of 1s; it would be acceptable to use the recurrence $|A_n| = |A_{n-1}| - |A_{n-2}|$ and the easy cases $|A_1| = 1$, $|A_2| = 0$.)

   (c) (4 points) $\begin{vmatrix} 1 & 2 & 1 \\ 3 & 1 & 3 \\ 1 & -2 & 1 \end{vmatrix}$

   Solution: There is a repeated column, so the matrix is singular and the determinant is $\boxed{0}$.

This page is a continuation of problem 1.

(d) (4 points) $A = \begin{pmatrix} 0 & 3 & 0 \\ 2 & 0 & 2 \\ -2 & 0 & 2 \end{pmatrix}$. What is $|A^{-1}|$?

Solution: In general, $|A^{-1}| = \frac{1}{|A|}$. Using the cofactor formula,

$$|A| = -3 \begin{vmatrix} 2 & 2 \\ -2 & 2 \end{vmatrix} = -3(4 - (-4)) = -24$$

Therefore $|A^{-1}| = \boxed{-\frac{1}{24}}$.

(e) (4 points) A is invertible. What is $|A^T(A^{-1})^2A^T|$?

Solution:

$$|A^T(A^{-1})^2A^T| = |A^T||(A^{-1})^2||A^T| = |A||A^{-1}|^2|A| = |A||A|^{-2}|A| = \boxed{1}$$

(f) (4 points) Q is orthogonal. What is $|Q^{2006}|$?

Solution: Since $Q^TQ = I$, $|Q|^2 = 1$, and

$$|Q^{2006}| = |Q|^{2006} = (|Q|^2)^{1003} = 1^{1003} = \boxed{1}.$$

Note: This works because 2006 *is even. The determinant of* Q^{2005}*, on the other hand, could be* 1 *or* -1.

2. Consider the 3×4 matrix

$$A = \begin{pmatrix} 1 & 1 & 2 & 3 \\ 1 & 2 & 3 & 1 \\ 0 & -1 & -1 & 2 \end{pmatrix}.$$

In this problem you will need to find bases for the four subspaces of A and solve a linear system $A\mathbf{x} = \mathbf{b}$.

(a) (4 points) Which subspaces are the same for A and its reduced row echelon matrix R? **Circle the statement(s) that are true.**

i. $C(A) = C(R)$

ii. $\boxed{N(A) = N(R)}$

iii. $\boxed{C(A^T) = C(R^T)}$

iv. $N(A^T) = N(R^T)$

Solution: Since R is obtained from A by row operations, the *row spaces* are the same, as are their orthogonal complements, the *null spaces*.

(b) (5 points) Compute the rank of A and the dimensions of the four subspaces. Put your answers in this table:

rank(A)	$\dim C(A)$	$\dim N(A)$	$\dim C(A^T)$	$\dim N(A^T)$
2	2	2	2	1

Solution: The rank and the size of A (3×4 in this case) determine all of the dimensions. We can find the rank from the reduced row echelon form:

$$A = \begin{pmatrix} 1 & 1 & 2 & 3 \\ 1 & 2 & 3 & 1 \\ 0 & -1 & -1 & 2 \end{pmatrix}$$

Subtract row 1 from row 2:

$$\begin{pmatrix} 1 & 1 & 2 & 3 \\ 0 & 1 & 1 & -2 \\ 0 & -1 & -1 & 2 \end{pmatrix}$$

Add row 2 to row 3:

$$\begin{pmatrix} 1 & 1 & 2 & 3 \\ 0 & 1 & 1 & -2 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

And finally subtract row 2 from row 1:

$$R = \begin{pmatrix} 1 & 0 & 1 & 5 \\ 0 & 1 & 1 & -2 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

We have two pivots, so $\text{rank}(A) = \dim C(A) = \dim C(A^T) = 2$. Also $\dim N(A) = (n - r) = 4 - 2 = 2$ and $\dim N(A^T) = (m - r) = 3 - 2 = 1$.

This page is a continuation of problem 2.

$$A = \begin{pmatrix} 1 & 1 & 2 & 3 \\ 1 & 2 & 3 & 1 \\ 0 & -1 & -1 & 2 \end{pmatrix}.$$

(c) (4 points) Find a basis for $C(A)$.

Solution: The first two columns of A are the pivot columns, so they form a basis for $C(A)$:

$$\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \quad \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$$

(d) (4 points) Find a basis for $N(A)$.

Solution: The two special solutions are a basis for $N(A)$, and can be read off from the reduced row echelon matrix R (using $N = \left(\begin{smallmatrix} -F \\ I \end{smallmatrix}\right)$):

$$\begin{pmatrix} -1 \\ -1 \\ 1 \\ 0 \end{pmatrix} \quad \begin{pmatrix} -5 \\ 2 \\ 0 \\ 1 \end{pmatrix}$$

(e) (4 points) Find a basis for $C(A^T)$.

Solution: The two nonzero rows of R are a basis for $C(A^T)$:

$$\begin{pmatrix} 1 \\ 0 \\ 1 \\ 5 \end{pmatrix} \quad \begin{pmatrix} 0 \\ 1 \\ 1 \\ -2 \end{pmatrix}$$

(f) (4 points) Find a basis for $N(A^T)$.

Solution: Observe that the first row of A is the sum of the last two rows, thus

$$\begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}$$

is a vector in $N(A^T)$. Since $N(A^T)$ is one-dimensional, this is a basis. (We could also find the elimination matrix E, which has this vector as its third row.)

This page is a continuation of problem 2.

$$A = \begin{pmatrix} 1 & 1 & 2 & 3 \\ 1 & 2 & 3 & 1 \\ 0 & -1 & -1 & 2 \end{pmatrix}.$$

(g) (3 points) Find the **general solution** to $A\mathbf{x} = \begin{pmatrix} 2 \\ 2 \\ 0 \end{pmatrix}$. If there are no solutions at all, write "no solutions".

Solution: Since $\begin{pmatrix} 2 \\ 2 \\ 0 \end{pmatrix}$ is 2 times the first column of A, a particular solution is simply

$$\mathbf{x}_p = \begin{pmatrix} 2 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

The general solution is the sum of this particular solution and a vector in the null space, i.e.

$$\mathbf{x} = \begin{pmatrix} 2 \\ 0 \\ 0 \\ 0 \end{pmatrix} + c_1 \begin{pmatrix} -1 \\ -1 \\ 1 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} -5 \\ 2 \\ 0 \\ 1 \end{pmatrix}$$

3. Let $A = \begin{pmatrix} -2 & 4 \\ 1 & 1 \\ 2 & -1 \end{pmatrix}$.

   (a) (10 points) Use the Gram-Schmidt algorithm to find an orthonormal basis $\mathbf{q}_1, \mathbf{q}_2$ for the column space of A.

   Solution: To find $\mathbf{q}_1$ we need only normalize the length of the first column $\mathbf{a}_1$:

$$\mathbf{q}_1 = \frac{\mathbf{a}_1}{\|\mathbf{a}_1\|} = \frac{1}{3}\begin{pmatrix} -2 \\ 1 \\ 2 \end{pmatrix}$$

   To find a vector $\mathbf{A}_2$ that is orthogonal to $\mathbf{q}_1$, we subtract the projection of $\mathbf{a}_2$ onto $\mathbf{q}_1$ from $\mathbf{a}_2$:

$$\mathbf{A}_2 = \mathbf{a}_2 - (\mathbf{q}_1^T\mathbf{a}_2)\mathbf{q}_1 = \begin{pmatrix} 4 \\ 1 \\ -1 \end{pmatrix} - \frac{1}{9}(-8+1-2)\begin{pmatrix} -2 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}$$

   Finally we normalize the length of $\mathbf{A}_2$ to obtain $\mathbf{q}_2$:

$$\mathbf{q}_2 = \frac{\mathbf{A}_2}{\|\mathbf{A}_2\|} = \frac{1}{3}\begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}$$

   (b) (10 points) There is no solution to $A\mathbf{x} = \mathbf{b}$ when $\mathbf{b} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$. Find the best approximate solution $\hat{\mathbf{x}}$, which minimizes $\|A\hat{\mathbf{x}} - \mathbf{b}\|$.

   Solution: We need to solve the equation $A^TA\hat{\mathbf{x}} = A^T\mathbf{b}$. We compute

$$A^TA = \begin{pmatrix} 9 & -9 \\ -9 & 18 \end{pmatrix} \quad A^T\mathbf{b} = \begin{pmatrix} -2 \\ 4 \end{pmatrix}$$

   This system can be solved in several ways. For instance, we can apply the formula for the inverse of a 2×2 matrix to obtain:

$$\hat{\mathbf{x}} = (A^TA)^{-1}\mathbf{b} = \begin{pmatrix} 9 & -9 \\ -9 & 18 \end{pmatrix}^{-1}\begin{pmatrix} -2 \\ 4 \end{pmatrix} = \frac{1}{9}\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} -2 \\ 4 \end{pmatrix} = \begin{pmatrix} 0 \\ \frac{2}{9} \end{pmatrix}$$

4. Let V be the subspace of $\mathbb{R}^4$ spanned by the vectors $\begin{pmatrix}1\\0\\0\\1\end{pmatrix}$, $\begin{pmatrix}0\\1\\1\\0\end{pmatrix}$, and $\begin{pmatrix}1\\1\\1\\1\end{pmatrix}$.

   (a) (5 points) What is the dimension of V?

   Solution: The first two vectors are orthogonal and hence independent, so the dimension is at least 2. Since the third vector is the sum of the first two, it is already in the plane they span, and $\dim V = 2$.

   (b) (5 points) Find a basis for V.

   Solution: The first two vectors are independent, and thus form a basis for V:

$$\begin{pmatrix}1\\0\\0\\1\end{pmatrix} \quad \begin{pmatrix}0\\1\\1\\0\end{pmatrix}$$

   (c) (5 points) Find a basis for the orthogonal complement $V^\perp$.

   Solution: Notice that V is the row space of the matrix

$$B = \begin{pmatrix}1 & 0 & 0 & 1\\ 0 & 1 & 1 & 0\end{pmatrix}$$

and therefore $V^\perp = C(B^T)^\perp = N(B)$. The matrix B is already in reduced row echelon form, so we can find the null space matrix easily.

$$N = \begin{pmatrix}-F\\ I\end{pmatrix} = \begin{pmatrix}0 & -1\\ -1 & 0\\ 1 & 0\\ 0 & 1\end{pmatrix}$$

The columns of N are therefore a basis for $V^\perp$.

$$\begin{pmatrix}0\\-1\\1\\0\end{pmatrix} \quad \begin{pmatrix}-1\\0\\0\\1\end{pmatrix}$$

This page continues problem 4; V is the span of $\begin{pmatrix}1\\0\\0\\1\end{pmatrix}$, $\begin{pmatrix}0\\1\\1\\0\end{pmatrix}$, and $\begin{pmatrix}1\\1\\1\\1\end{pmatrix}$.

(d) (5 points) Find the projection of $\mathbf{b} = \begin{pmatrix}0\\2\\0\\0\end{pmatrix}$ onto V.

Solution: Notice that

$$\begin{pmatrix}0\\2\\0\\0\end{pmatrix} = \begin{pmatrix}0\\1\\1\\0\end{pmatrix} + \begin{pmatrix}0\\1\\-1\\0\end{pmatrix},$$

and that this expresses $\mathbf{b}$ as a sum of a vector in V and a vector in $V^{\perp}$. The projection of $\mathbf{b}$ onto V is therefore

$$\mathbf{p} = \begin{pmatrix}0\\1\\1\\0\end{pmatrix}$$

since this makes $\mathbf{e} = (\mathbf{b} - \mathbf{p})$ orthogonal to V. (We could also use the formula for the projection onto the column space of a matrix to get the same result.)

(e) (5 points) Find the projection of $\mathbf{b} = \begin{pmatrix}0\\2\\0\\0\end{pmatrix}$ onto $V^{\perp}$.

Solution: The projection onto $V^{\perp}$ is just the error when we project $\mathbf{b}$ onto V, i.e.

$$\mathbf{e} = \mathbf{b} - \mathbf{p} = \begin{pmatrix}0\\1\\-1\\0\end{pmatrix}$$