1. Suppose A is a 3×3 matrix with the property that (row 3) = 5(row 1) + 11(row 2).

   (a) (5 points) Explain why there is no solution to $A\mathbf{x} = \begin{pmatrix}1\\1\\1\end{pmatrix}$

   Solution: Since (row 3) = 5(row 1) + 11(row 2), each column of A has the form

   $$\begin{pmatrix}a_1\\a_2\\a_3\end{pmatrix}$$

   where $a_3 = 5a_1 + 11a_2$. The product $A\mathbf{x}$ is a linear combination of these columns, so it also has this property.

   But $\mathbf{b} = \begin{pmatrix}1\\1\\1\end{pmatrix}$ does not have this property, so it is not a linear combination of the columns of A. In other words, $A\mathbf{x} = \begin{pmatrix}1\\1\\1\end{pmatrix}$ has no solution.

   (b) (5 points) What condition on b_1, b_2, b_3 is necessary for $A\mathbf{x} = \begin{pmatrix}b_1\\b_2\\b_3\end{pmatrix}$ to have a solution?

   Solution: As we saw in part (a), the vector $\mathbf{b}$ must satisfy $b_3 = 5b_1 + 11b_2$, or else it cannot be obtained as a linear combination of the columns of A.

   (c) (5 points) Explain why A is not invertible.

   Solution: If A were invertible, then $A\mathbf{x} = \mathbf{b}$ would have a solution for every vector $\mathbf{b}$ (in fact, the solution would be $\mathbf{x} = A^{-1}\mathbf{b}$). But we have already seen that $A\mathbf{x} = \mathbf{b}$ may not have a solution, so A cannot be invertible.

(d) (5 points) Given that (row 3) = 5(row 1) + 11(row 2), does

$$A\mathbf{x} = \begin{pmatrix} 1 \\ 1 \\ 16 \end{pmatrix}$$

necessarily have a solution? If so, explain why. If not, give an example of a matrix A meeting the condition but for which there is no solution.

Solution: It is possible that $Ax = \begin{pmatrix} 1 \\ 1 \\ 16 \end{pmatrix}$ has no solution. For example, if $A = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 5 & 5 & 5 \end{pmatrix}$, then A satisfies the row condition, but for any vector $\mathbf{x}$, the second component of $A\mathbf{x}$ is zero. Thus there is no solution to $Ax = \begin{pmatrix} 1 \\ 1 \\ 16 \end{pmatrix}$.

2. The three parts of this question concern the 3×3 matrix

$$A = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & -1 \\ -2 & 3 & -9 \end{pmatrix}$$

(The matrix is repeated at the top of each page that is part of this question.)

(a) (10 points) Find the matrices L (lower triangular) and U (upper triangular) in the decomposition $A = LU$.

Solution: As we apply row operations to A, we get the entries of L.

$$A = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & -1 \\ -2 & 3 & -9 \end{pmatrix} \qquad L = \begin{pmatrix} 1 & 0 & 0 \\ ? & 1 & 0 \\ ? & ? & 1 \end{pmatrix}$$

Subtract 1 times row 1 from row 2.

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -2 \\ -2 & 3 & -9 \end{pmatrix} \qquad L = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ ? & ? & 1 \end{pmatrix}$$

Subtract -2 times row 1 from row 3.

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -2 \\ 0 & 3 & -7 \end{pmatrix} \qquad L = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ -2 & ? & 1 \end{pmatrix}$$

Subtract 3 times row 2 from row 3.

$$U = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & -1 \end{pmatrix} \qquad L = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ -2 & 3 & 1 \end{pmatrix}$$

$$A = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & -1 \\ -2 & 3 & -9 \end{pmatrix}$$

(b) (5 points) What is the rank of A?

Solution: Elimination on A (when we computed the LU decomposition) found 3 pivots, so $\mathbf{Rank(A) = 3}$.

(c) (5 points) Describe the null space $N(A)$.

Solution: Since A is invertible (equivalently, it has 3 pivots, there are no free columns), the only solution to $A\mathbf{x} = \mathbf{0}$ is $\mathbf{x} = \mathbf{0}$.

Therefore, the null space is **just the zero vector,** $N(A) = \mathbf{0}$.

3. The four parts of this question concern the matrix

$$A = \begin{pmatrix} -1 & 3 & 11 & 7 \\ 1 & 6 & 7 & 20 \end{pmatrix}$$

(The matrix is repeated at the top of each page that is part of this question.)

(a) (5 points) Describe the column space $C(A)$. (Don't just state the definition, describe it for this particular matrix.)

Solution: The column space is a subspace of $\mathbb{R}^2$, consisting of the linear combinations of the four columns of A. But the first two columns of A, i.e. $\mathbf{v}_1 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$ and $\mathbf{v}_2 = \begin{pmatrix} 3 \\ 6 \end{pmatrix}$, are two vectors that point in different directions, so their linear combinations fill all of $\mathbb{R}^2$. The rest of the columns are superfluous.
The column space is $C(A) = \mathbb{R}^2$.

(b) (10 points) Find the reduced row echelon form R for A.

Solution: Add row 1 to row 2.

$$\begin{pmatrix} -1 & 3 & 11 & 7 \\ 0 & 9 & 18 & 27 \end{pmatrix}$$

This is echelon form, with two pivots (-1 and 9). Columns 3 and 4 are free. To proceed to reduced row echelon form, we eliminate above the pivot 9.
Subtract $\frac{1}{3}$ times row 2 from row 1.

$$\begin{pmatrix} -1 & 0 & 5 & -2 \\ 0 & 9 & 18 & 27 \end{pmatrix}$$

Finally, divide each row by its pivot (i.e. divide row 1 by -1, and divide row 2 by 9).

$$R = \begin{pmatrix} 1 & 0 & -5 & 2 \\ 0 & 1 & 2 & 3 \end{pmatrix}$$

$$A = \begin{pmatrix} -1 & 3 & 11 & 7 \\ 1 & 6 & 7 & 20 \end{pmatrix}$$

(c) (5 points) What is the rank of A?

Solution: We found 2 pivots while computing the (reduced row) echelon form, so $\mathbf{Rank(A) = 2}$.

(d) (10 points) Describe the null space $N(A)$. (Describe it as the set of linear combinations of a few vectors.)

Solution: When the pivot columns are at the left and there are no zero rows (as is the case here), the reduced row echelon form is

$$R = \begin{pmatrix} I & F \end{pmatrix}$$

and the null space matrix, whose columns are the special solutions, is

$$N = \begin{pmatrix} -F \\ I \end{pmatrix}.$$

Looking at the reduced row echelon form from part (a), we have

$$F = \begin{pmatrix} -5 & 2 \\ 2 & 3 \end{pmatrix} \quad N = \begin{pmatrix} 5 & -2 \\ -2 & -3 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Therefore, the null space $N(A)$ consists of all linear combinations of the columns of N (the two special solutions):

$$c_1 \begin{pmatrix} 5 \\ -2 \\ 1 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} -2 \\ -3 \\ 0 \\ 1 \end{pmatrix}$$

4. Let $\mathbf{M}_{2\times 2}$ be the vector space of all 2×2 matrices. An element of $\mathbf{M}_{2\times 2}$ looks like:

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

Determine whether or not each of the following subsets of $\mathbf{M}_{2\times 2}$ is a subspace.

(a) (5 points) The matrices with $a_{11} \neq 0$.

Solution: This **is NOT a subspace**, because it does not contain the zero matrix (which is the zero vector in $\mathbf{M}_{2\times 2}$).

(b) (5 points) The matrices with $a_{11} = 0$.

Solution: This **is a subspace**, because the linear condition $a_{11} = 0$ is preserved when taking sums or scalar multiples of matrices.

(c) (5 points) The matrices with at least one entry equal to zero.

Solution: This **is NOT a subspace**, because it is not closed under vector addition. For example, the matrices $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ are in the set, but their sum $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ is not.

(d) (5 points) The matrices with $a_{11} \geq a_{12}$.

Solution: This **is NOT a subspace**, because it is not closed under scalar multiplication. For example, the set contains the identity matrix I (because $1 \geq 0$) but not the scalar multiple $-I = (-1)\,I$ (because $-1 < 0$).