1. Let $\mathbf{M}_{2\times 2}$ be the vector space of all 2×2 matrices. An element of $\mathbf{M}_{2\times 2}$ looks like:

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

Determine whether or not each of the following subsets of $\mathbf{M}_{2\times 2}$ is a subspace.

(a) (5 points) The matrices with $a_{11} = 0$.

Solution: This **is a subspace**, because the linear condition $a_{11} = 0$ is preserved when taking sums or scalar multiples of matrices.

(b) (5 points) The matrices with $a_{11} \neq 0$.

Solution: This **is NOT a subspace**, because it does not contain the zero matrix (which is the zero vector in $\mathbf{M}_{2\times 2}$).

(c) (5 points) The matrices with $a_{11} \geq a_{12}$.

Solution: This **is NOT a subspace**, because it is not closed under scalar multiplication. For example, the set contains the identity matrix I (because $1 \geq 0$) but not the scalar multiple $-I = (-1)\,I$ (because $-1 < 0$).

(d) (5 points) The matrices with at least one entry equal to zero.

Solution: This **is NOT a subspace**, because it is not closed under vector addition. For example, the matrices $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ are in the set, but their sum $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ is not.

2. The three parts of this question concern the 3×3 matrix

$$A = \begin{pmatrix} 1 & 0 & -1 \\ -1 & 1 & 3 \\ 2 & 4 & 5 \end{pmatrix}$$

(The matrix is repeated at the top of each page that is part of this question.)

(a) (10 points) Find the matrices L (lower triangular) and U (upper triangular) in the decomposition $A = LU$.

Solution: As we apply row operations to A, we get the entries of L.

$$A = \begin{pmatrix} 1 & 0 & -1 \\ -1 & 1 & 3 \\ 2 & 4 & 5 \end{pmatrix} \quad L = \begin{pmatrix} 1 & 0 & 0 \\ ? & 1 & 0 \\ ? & ? & 1 \end{pmatrix}$$

Subtract -1 times row 1 from row 2.

$$\begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 2 & 4 & 5 \end{pmatrix} \quad L = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ ? & ? & 1 \end{pmatrix}$$

Subtract 2 times row 1 from row 3.

$$\begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 4 & 7 \end{pmatrix} \quad L = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 2 & ? & 1 \end{pmatrix}$$

Subtract 4 times row 2 from row 3.

$$U = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & -1 \end{pmatrix} \quad L = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 2 & 4 & 1 \end{pmatrix}$$

$$A = \begin{pmatrix} 1 & 0 & -1 \\ -1 & 1 & 3 \\ 2 & 4 & 5 \end{pmatrix}$$

(b) (5 points) What is the rank of A?

Solution: Elimination on A (when we computed the LU decomposition) found 3 pivots, so $\mathbf{Rank(A) = 3}$.

(c) (5 points) Describe the null space $N(A)$.

Solution: Since A is invertible (equivalently, it has 3 pivots, there are no free columns), the only solution to $A\mathbf{x} = \mathbf{0}$ is $\mathbf{x} = \mathbf{0}$.
Therefore, the null space is **just the zero vector,** $N(A) = \mathbf{0}$.

3. The four parts of this question concern the matrix

$$A = \begin{pmatrix} 1 & 2 & 10 & 5 \\ -1 & 0 & -6 & 1 \end{pmatrix}$$

(The matrix is repeated at the top of each page that is part of this question.)

(a) (5 points) Describe the column space $C(A)$. (Don't just state the definition, describe it for this particular matrix.)

Solution: The column space is a subspace of $\mathbb{R}^2$, consisting of the linear combinations of the four columns of A. But the first two columns of A, i.e. $\mathbf{v}_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ and $\mathbf{v}_2 = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$, are two vectors that point in different directions, so their linear combinations fill all of $\mathbb{R}^2$. The rest of the columns are superfluous.
The column space is $C(A) = \mathbb{R}^2$.

(b) (10 points) Find the reduced row echelon form R for A.

Solution: Add row 1 to row 2.

$$\begin{pmatrix} 1 & 2 & 10 & 5 \\ 0 & 2 & 4 & 6 \end{pmatrix}$$

This is echelon form, with two pivots (1 and 2). Columns 3 and 4 are free. To proceed to reduced row echelon form, we eliminate above the pivot 2.
Subtract row 2 from row 1.

$$\begin{pmatrix} 1 & 0 & 6 & -1 \\ 0 & 2 & 4 & 6 \end{pmatrix}$$

Finally, divide each row by its pivot (i.e. divide row 2 by 2).

$$R = \begin{pmatrix} 1 & 0 & 6 & -1 \\ 0 & 1 & 2 & 3 \end{pmatrix}$$

$$A = \begin{pmatrix} 1 & 2 & 10 & 5 \\ -1 & 0 & -6 & 1 \end{pmatrix}$$

(c) (5 points) What is the rank of A?

> **Solution:** We found 2 pivots while computing the (reduced row) echelon form, so $\mathbf{Rank(A) = 2}$.

(d) (10 points) Describe the null space $N(A)$. (Describe it as the set of linear combinations of a few vectors.)

> **Solution:** When the pivot columns are at the left and there are no zero rows (as is the case here), the reduced row echelon form is
>
> $$R = \begin{pmatrix} I & F \end{pmatrix}$$
>
> and the null space matrix, whose columns are the special solutions, is
>
> $$N = \begin{pmatrix} -F \\ I \end{pmatrix}.$$
>
> Looking at the reduced row echelon form from part (a), we have
>
> $$F = \begin{pmatrix} 6 & -1 \\ 2 & 3 \end{pmatrix} \quad N = \begin{pmatrix} -6 & 1 \\ -2 & -3 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}$$
>
> Therefore, the null space $N(A)$ consists of all linear combinations of the columns of N (the two special solutions):
>
> $$c_1 \begin{pmatrix} -6 \\ -2 \\ 1 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ -3 \\ 0 \\ 1 \end{pmatrix}$$

4. Suppose A is a 3×3 matrix with the property that (row 1) = 2(row 2) + 17(row 3).

   (a) (5 points) Explain why there is no solution to $A\mathbf{x} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

   Solution: Since (row 1) = 2(row 2) + 17(row 3), each column of A has the form

   $$\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$$

   where $a_1 = 2a_2 + 17a_3$. The product $A\mathbf{x}$ is a linear combination of these columns, so it also has this property.

   But $\mathbf{b} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ does not have this property, so it is not a linear combination of the columns of A. In other words, $A\mathbf{x} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ has no solution.

   (b) (5 points) What condition on b_1, b_2, b_3 is necessary for $A\mathbf{x} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$ to have a solution?

   Solution: As we saw in part (a), the vector $\mathbf{b}$ must satisfy $b_1 = 2b_2 + 17b_3$, or else it cannot be obtained as a linear combination of the columns of A.

   (c) (5 points) Explain why A is not invertible.

   Solution: If A were invertible, then $A\mathbf{x} = \mathbf{b}$ would have a solution for every vector $\mathbf{b}$ (in fact, the solution would be $\mathbf{x} = A^{-1}\mathbf{b}$). But we have already seen that $A\mathbf{x} = \mathbf{b}$ may not have a solution, so A cannot be invertible.

(d) (5 points) Given that (row 1) = 2(row 2) + 17(row 3), does

$$A\mathbf{x} = \begin{pmatrix} 19 \\ 1 \\ 1 \end{pmatrix}$$

necessarily have a solution? If so, explain why. If not, give an example of a matrix A meeting the condition but for which there is no solution.

Solution: It is possible that $Ax = \begin{pmatrix} 19 \\ 1 \\ 1 \end{pmatrix}$ has no solution. For example, if $A = \begin{pmatrix} 17 & 17 & 17 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \end{pmatrix}$, then A satisfies the row condition, but for any vector $\mathbf{x}$, the second component of $A\mathbf{x}$ is zero. Thus there is no solution to $Ax = \begin{pmatrix} 19 \\ 1 \\ 1 \end{pmatrix}$.