Groups of Möbius Transformations

Final Project Topic Suggestions

These are just suggestions – you are welcome to pursue a topic of your own invention.

Computer-Related Projects:

(1) **Schottky Plotting** Following the suggestions in Indra (or elsewhere), develop and implement an algorithm to plot the limit set of a Schottky group. This can be done within a mathematical programming environment like MATLAB or Mathematica, or as a stand-alone program. Your program should also be able to plot the orbits of structures related to the Schottky group, like the fundamental domain and its boundary circles.

(2) **Naive Discreteness Test** Consider a family of subgroups of $PSL_2(\mathbb{C})$, like the Maskit slice, and write a program that tries to determine which groups in the family are discrete. You might try looking for nearly-elliptic elements, which are a certain indication of nearby indiscreteness. A more high-tech solution would apply rigorous tests like the Jorgensen inequality to pairs of elements. Your program should be able to draw a picture for any one-parameter family of two-generator groups that is specified by giving formulas for the generating matrices. Test your program on the Maskit slice.

(3) **Maskit Boundary** Indra (pp 282 - 314) contains a sketch of an algorithm to trace the boundary of the set of discrete groups in the Maskit slice. Roughly speaking, this algorithm takes the following form: For any rational number p/q, there is an associated word in the generators a, b which becomes parabolic at some point on the boundary of the discreteness locus. The trace of this word is a polynomial $P_{p/q}(\mu)$ in the parameter μ, so this point can be located as a root of $P_{p/q}(\mu) - 2$. By selecting an appropriate (and large) set of rational numbers in $[0, 1]$, one can find the right roots of the associated polynomials with newton's method, and connect the roots to draw a picture of the boundary. Implement this algorithm, and if possible, compare your output with Dave Wright's program Kleinian.

(4) **Close Returns and Discontinuity** Given two generators for a suitable Kleinian group (one of the quasifuchsian groups from Indra would be a good start), write a program that tries to locate the domain of discontinuity as follows: Staring with a point $z \in \mathbb{C}$, compute its image under a few thousand short words in the generators, and let $\delta(z)$ denote the minimum distance between z and one of these images. The function $\delta(z)$ should get very small as z approaches the limit set, so one should be able to see Ω if a picture is drawn in which z is colored according to $\delta(z)$.

Paper-and-pencil Projects:

(1) **Elementary Groups** Read about the classification of elementary Kleinian groups in Maskit, Beardon, or a similar reference. Explain several aspects of this classification, such as:
 (a) A non-elementary group contains a loxodromic element
 (b) Elementary groups are virtually abelian
 (c) The topology of the quotient of the domain of discontinuity

(d) Flexibility or rigidity: For a given elementary group $\Gamma \subset PSL_2(\mathbb{C})$, are there isomorphic Kleinian groups that are not conjugate?
(e) Can an elementary Kleinian group be abstractly isomorphic to a non-elementary Kleinian group?

(2) **Relation to Hyperbolic Geometry** Explain the relationship between the action of $PSL_2(\mathbb{C})$ on $\hat{\mathbb{C}}$ by Möbius transformations and on $\mathbb{H}^3$ by isometries. Give formulas for the latter in at least one model of $\mathbb{H}^3$. Show that a subgroup of $PSL_2(\mathbb{C})$ is discrete if and only if it acts discontinuously on $\mathbb{H}^3$. Characterize the partition $\hat{\mathbb{C}} = \Lambda \cup \Omega$ for a Kleinian group Γ using the action on $\mathbb{H}^3$.

(3) **Fundamental Domains** Following the discussion in §9.5 - 9.9 in Beardon, explain the Ford construction of a fundamental domain for a Kleinian group with $\Omega \neq \emptyset$. This is one possible answer to the question of how to find a fundamental domain for a Kleinian group's action on its domain of discontinuity. Explain Poincaré's polyhedron theorem as a partial converse, allowing one to sometimes determine when a region is a fundamental domain for the group generated by a set of Möbius transformations acting on its domain of discontinuity.

(4) **Differential Geometry of Parameter Spaces** Use the action of $PSL_2(\mathbb{C})$ on $\hat{\mathbb{C}}$ to describe the geometry of the spaces of certain geometric objects related to Möbius geometry; for example, the space of
(a) round circles in $\hat{\mathbb{C}}$, or round disks
(b) pointed disks (i.e. pairs (D, z) where D is an open round disk and $z \in D$)
(c) pairs of distinct circles in $\hat{\mathbb{C}}$
(d) pencils of circles in $\hat{\mathbb{C}}$
(e) triples or quads of points
(f) half-turns in $PSL_2(\mathbb{C})$
(g) pairs of half-turns that generate a group isomorphic to $(\mathbb{Z}/2) \times (\mathbb{Z}/2)$
Each of these spaces has a natural action of $PSL_2(\mathbb{C})$, and some have natural compactifications – for example, a point can be seen as a limit of circles whose radii approach zero. Describe these in as much geometric and topological detail as possible.

(5) **Kleinian Groups as Subsets** $PSL_2(\mathbb{C})$ is homeomorphic to $S^3 \times R^3$ (prove this), and its double-cover $SL_2(C)$ is a subset of $\mathbb{C}^4$; in either description, one can think of a Kleinian group as a discrete scattering of points in a noncompact manifold. As in Beardon §5.3 or Maskit §II.B - II.C, discuss some results about the distribution of the elements of a Kleinian group. In particular, give a detailed proof of theorem 5.3.13 from Beardon, and of the Jorgensen inequality (Maskit §II.C.7).

(6) **Möbius Transformations in Many Dimensions** As in chapter 3 of Beardon, describe the group of Möbius transformations acting on $S^n = \hat{\mathbb{R}}^n$. Generalize elementary geometric facts related to $PSL_2(\mathbb{C})$ to this context, including:
(a) Spheres map to spheres
(b) Conformality
(c) Preservation of the cross-ratio
Most importantly, emphasize the great difference between Möbius geometry in $\hat{\mathbb{C}}$ and in higher dimensions by explaining and proving Liouville's theorem: A conformal

map of an open set in S^n, $n \geq 3$, is the restriction of a Möbius transformation (See Beardon chapter 3 for references). Note that this fails dramatically for $n = 2$.

(7) **Congruence Subgroups of** $PSL_2(\mathbb{Z})$ Analyze the action of $\Gamma(n) \subset PSL_2(\mathbb{Z})$ on the upper-half plane $\mathbb{H}$ for several small values of n. Describe a fundamental domain and the quotient $X(n) = \mathbb{H}/\Gamma(n)$. Use your examples to illustrate some general results about $\Gamma(n)$, like formulas for the genus, index, or number of cusps. Explain a general procedure that produces a fundamental domain for $\Gamma(n)$.

(8) **Bending** If you have some background in 2-dimensional hyperbolic geometry, you know that a compact surface Σ_g of genus $g > 1$ has many hyperbolic metrics, and that each gives rise to a representation $\rho : \pi_1(\Sigma_g) \to PSL_2(\mathbb{R})$ whose image is a Fuchsian group Γ acting on $\mathbb{H} = \tilde{\Sigma}_g$. Considering $PSL_2(\mathbb{R}) \subset PSL_2(\mathbb{C})$, Γ is also a Kleinian group, whose domain of discontinuity is $\mathbb{H} \cup \bar{\mathbb{H}}$ and whose limit set is $\hat{\mathbb{R}}$. Given a simple closed geodesic C of some hyperbolic metric on Σ_g, it is possible to deform the group Γ by "bending" the representation around α as follows: Suppose $\Sigma_g - \alpha$ is not disconnected, with connected components A and B. Then $\pi_1(\Sigma_g) = \pi_1(A) \star_\gamma \pi_1(B)$, where $a \in \pi_1(\Sigma_g)$ is a representative of the free homotopy class of C. Let $R_\theta \in PSL_2(\mathbb{C})$ be an elliptic element with rotation angle θ commuting with γ. The deformed representation is defined as:

$$\rho_\theta(\delta) = \begin{cases} \rho(\delta) & \text{if } \delta \in \pi_1(A) \\ R_\theta \rho(\delta) R_\theta^{-1} & \text{if } \delta \in \pi_1(B) \end{cases}$$

By the free product description of $\pi_1(\Sigma_g)$, this uniquely defines a representation $\rho_\theta : \pi_1(\Sigma_g) \to PSL_2(\mathbb{C})$. Fill in the details of this construction, and describe the bending process geometrically. Draw pictures of the limit set for several groups in a family ρ_θ.