Groups of Möbius Transformations

Homework 4

Note: This is a bit shorter than the previous assignments, as you are probably busy reading for your final project.

(1) (*Convergence of powers*) Let $\gamma \in PSL_2(\mathbb{C})$, $\gamma \neq I$, which as a Möbius transformation defines a holomorphic function $\gamma(z) : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$. By analyzing cases for the various types of Möbius transformations, show that there is a dichotomy in the behavior of the powers of γ, in that exactly one of the following holds:

(a) The set of powers $\{\gamma^i\}$ has an accumulation point in $PSL_2(\mathbb{C})$. (An accumulation point is a limit point of a convergent subsequence, so this condition is satisfied exactly when the sequence of powers $\{\gamma^i\}$ has a convergent subsequence. The constant sequence is convergent!)

(b) There are points $z_-, z_+ \in \hat{\mathbb{C}}$ and a sequence of distinct powers $\{\gamma^{n_i}\}$ such that the functions $\gamma^{n_i}(z)$ converge to the constant function z_+ on compact subsets of $\hat{\mathbb{C}} - \{z_-\}$.

(*Hint: This is really just a restatement of the classification of dynamics of a single Möbius transformation. See Ahlfors for a discussion of convergence for holomorphic functions.*)

(2) (*Convergence lemma*) Prove the following stronger version of the convergence lemma for sequences in $PSL_2(\mathbb{C})$:

Lemma 0.1. *Let $\{\gamma_i\} \subset PSL_2(\mathbb{C})$ be a sequence with no accumulation points (in particular, the sequence contains infinitely many distinct elements). Then there exists a subsequence γ_{n_i} and points $z_-, z_+ \in \hat{\mathbb{C}}$ such that the sequence of functions $\gamma_{n_i}(z)$ converges to the constant function z_+ uniformly on compact subsets of $\hat{\mathbb{C}} - \{z_-\}$.*

Recall that in class we proved a similar but weaker convergence lemma, showing that if $\{\gamma_i\} \subset \Gamma$ is a sequence of distinct elements in a Kleinian group, and if $\gamma_i(z) \to w$, then $w \in \Lambda_\Gamma$.

(3) (Λ *for subgroups*) Let Γ be a nonelementary Kleinian group, and $\Gamma' \subset \Gamma$ a subgroup. Observe that Γ' is Kleinian.

(a) Show that $\Lambda_{\Gamma'} \subset \Lambda_\Gamma$, and that in general these two sets differ.

(b) If Γ' has finite index in Γ, then $\Lambda_{\Gamma'} = \Lambda_\Gamma$.

(c) If Γ' is *normal* in Γ, then $\Lambda_{\Gamma'} = \Lambda_\Gamma$. (*Hint:* $\Lambda_{\Gamma'} \neq \emptyset$.)

(4) In lecture, we showed that $PSL_2(\mathbb{Z})$ has limit set $\hat{\mathbb{R}}$ by showing that the set of loxodromic fixed points of elements of $PSL_2(\mathbb{Z})$ is a dense subset of $\mathbb{R}$. Let $\mathscr{O} = \{m + ni \mid m, n \in \mathbb{Z}\} \subset \mathbb{C}$ be the set of Gaussian integers. Show that $PSL_2(\mathscr{O})$ is a Kleinian group, and that its limit set is $\Lambda = \hat{\mathbb{C}}$.

(5) We have defined the limit set as the closure of the set of loxodromic fixed points. Show that this definition is not well-adapted to subgroups of $PSL_2(\mathbb{C})$ that are not Kleinian:

(a) Give an example of a finitely generated subgroup $\Gamma \subset PSL_2(\mathbb{C})$ whose limit set is empty, but which does not act discontinuously on any open set in $\hat{\mathbb{C}}$.

(b) Give an example of a subgroup $\Gamma \subset PSL_2(\mathbb{C})$ whose limit set Λ is uncountable, but which does not act discontinuously on $\Omega = \hat{\mathbb{C}} - \Lambda$. (*Hint: With no conditions on how big* Γ *can be, this is easy. It is also possible to choose* Γ *to be finitely generated.*)

(6) (Λ *often fractal*)

(a) Suppose Γ is a Kleinian group whose limit set Λ contains a differentiable arc A, i.e. there is a differentiable map $\alpha : (-1, 1) \to \hat{\mathbb{C}}$ whose image A is a subset of Λ. Show that Λ contains a round circle through a point in A.

(b) Conclude that if Λ is a Jordan curve (a continuous closed curve; formally, the image of a continuous injection $f : S^1 \to \hat{\mathbb{C}}$) which is differentiable on some open interval, then Λ is a round circle.