Groups of Möbius Transformations

Homework 3

Note: Problems are of widely varying difficulty, and long problems are often the easy ones!

(1) (*Identifying Loxodromics*)

(a) Let D be an open round disk in $\mathbb{C}$, and $\bar{D}$ its closure. Suppose $\gamma \in PSL_2(\mathbb{C})$ and $\gamma(\bar{D}) \subset D$. Show that γ is loxodromic (or hyperbolic) with at least one fixed point in D.

(b) Use this to conclude that the fixed points of the generators of a classical schottky group are arranged as we've assumed in class; i.e. a has one fixed point in the interior of D_A and the other in the interior of D_a, and similarly for b.

(2) Analyze how the cross ratio depends on the order of the four points. Specifically, let $\chi_0 = \chi(z_1, z_2, z_3, z_4)$, and $\sigma \in S_4$, the symmetric group on four letters. This means that $\sigma : \{1, 2, 3, 4\} \to \{1, 2, 3, 4\}$ is a bijection. Express $\chi(\sigma) = \chi(z_{\sigma(1)}, z_{\sigma(2)}, z_{\sigma(3)}, z_{\sigma(4)})$ in terms of χ_0. (*Hint: It would suffice to list the result for each permutation of four letters, but a pattern should emerge if you attempt this. How many different values occur?*)

(3) (*Half Turns*) An elliptic Möbius transformation of order 2 (i.e. $\gamma^2 = I$) is called a *half turn*. Recall that on homework 1, you proved that there are groups of half turns isomorphic to $(\mathbb{Z}/2) \times (\mathbb{Z}/2)$, and that all such groups are conjugate.

(a) Show that $\gamma \in PSL_2(\mathbb{C})$ is a half turn if and only if $\mathrm{tr}(\gamma) = 0$.

(b) Let $a, b \in PSL_2(\mathbb{C})$ be half turns. Show that $\Gamma = \langle a, b\rangle \subset PSL_2(\mathbb{C})$ is discrete and elementary.(*8/4/2002: Oops! This is not true. For example, any elliptic is the composition of two half turns.*)

(c) Let χ be the cross ratio of the set of fixed points of a and b, assuming they are distinct and ordering them in some way. Use χ to classify the groups that arise as $\langle a, b\rangle$, then analyze the special cases where a and b share fixed points. Show that χ "almost" determines the group up to conjugacy.

(4) Let Γ be a discrete, non-elementary Kleinian group, and Ω its domain of discontinuity. Show that if $\{0, \infty\} \subset \Omega$, then the four matrix coefficients of any element of Γ are of comparable size. More precisely, show that there is a constant $K > 0$ such that for all $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$,

$$|a| \leq K|b| \leq K^2|c| \leq K^3|d| \leq K^4|a|.$$

Give examples showing why this doesn't work if $\{0, \infty\} \cap \Lambda \neq \emptyset$. (Maskit)

(5) Construct Kleinian groups with elliptic elements such that

(a) neither

(b) one

(c) both

of the fixed points lie(s) in the limit set Λ. (*Hint: It is possible to do this by adding elliptic elements to examples considered in class.*) (Maskit)

(6) (*Anti-Schottky groups*) Let $\mathscr{C}$ be a collection of N disjoint, unnested circles in $\mathbb{C}$, and let $\Gamma_{\mathscr{C}}$ be the group generated by the inversions in these circles. Recall that the

inversion in a circle C is the unique anti-Möbius transformation that fixes each point of C and exchanges the inside and outside.

(a) Show that if N is even, then $\Gamma_{\mathscr{C}}$ has a classical schottky group of rank $N/2$ as a subgroup of finite index, where the generators of the schottky group pair circles in $\mathscr{C}$.

(b) What happens if N is odd? Draw a picture of the orbit of $\mathscr{C}$ under $\Gamma_{\mathscr{C}}$ for $N = 3$.

(7) (*Congruence subgroups*) Recall that the group $PSL_2(\mathbb{Z}) \subset PSL_2(\mathbb{C})$ was considered on homework 1 and 2, and that the Möbius transformations $T = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ and $T' = \left(\begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix}\right)$ generate $PSL_2(\mathbb{Z})$, which is evidently a discrete subgroup of $PSL_2(\mathbb{C})$. We will now investigate how this group provides a rich supply of non-elementary, torsion-free Kleinian groups.

(a) Show that $T = \left(\begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix}\right)$ and $S = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$ also generate $PSL_2(\mathbb{Z})$. Note that the latter is a half-turn.

(b) Show that $PSL_2(\mathbb{Z})$ also contains an elliptic element R of order 3. Furthermore, any elliptic element of $PSL_2(\mathbb{Z})$ is conjugate to either R or S, and in particular has order 2 or 3.

(c) For any integer $n > 0$, define

$$\Gamma(n) = \{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in PSL_2(\mathbb{Z}) \bigm| a, d \equiv 1 \bmod n,\ b, c \equiv 0 \bmod n \},$$

which is called the principal congruence subgroup of level n; in other words, $\Gamma(n)$ consists of those Möbius transformations that can be represented by integral matrices that are congruent to the identity mod n. Show that $\Gamma(n)$ is a normal subgroup of $PSL_2(\mathbb{Z})$, and that for $n > 1$, $\Gamma(n)$ is torsion-free and non-elementary.

(d) $\Gamma(1)$ is a shorthand notation for $PSL_2(\mathbb{Z})$. Show that $\Gamma(2)$ is generated by T^2 and T'^2, has finite index in $\Gamma(1)$, and that the quotient $G = \Gamma(1)/\Gamma(2)$ is isomorphic to S_3, the symmetric group on three letters.

(8) (*Trace Identities*) Because this problem is about traces, let's consider matrices in $SL_2(\mathbb{C})$ to avoid the ± 1 ambiguity. The corresponding results for $PSL_2(\mathbb{C})$ should be evident.

(a) Use the Cayley-Hamilton theorem (a matrix is a root of its own characteristic polynomial) to show that for any $A \in SL_2(\mathbb{C})$,

$$A^2 - (\mathrm{tr}A)A + 1 = 0.$$

(b) Use this identity to show that any two elements $A, B \in SL_2(\mathbb{C})$ satisfy

$$\mathrm{tr}(AB) + tr(AB^{-1}) = \mathrm{tr}(A)\mathrm{tr}(B).$$

(c) Suppose that $A, B \in SL_2(\mathbb{C})$ and $[A, B] = ABA^{-1}B^{-1}$ is parabolic. Let $x = \mathrm{tr}(A)$, $y = \mathrm{tr}(B)$, $z = \mathrm{tr}(C)$. Show that

$$x^2 + y^2 + z^2 = zyz.$$

(d) Show that for any $x, y \in \mathbb{C}$, there are matrices $A, B \in SL_2(\mathbb{C})$ with $[A, B]$ parabolic and $x = \mathrm{tr}(A)$, $y = \mathrm{tr}(B)$.

(e) (*Bonus*) Describe the manifold $\{(x, y, z) \mid x^2 + y^2 + z^2 = xyz\} \subset \mathbb{C}^3$.