Groups of Möbius Transformations

Homework 2

Note: Problems are of widely varying difficulty!

(1) (*The Cross Ratio*) Let z_1, z_2, z_3, z_4 be four distinct points in $\mathbb{C}$. Define

$$\chi(z_1, z_2, z_3, z_4) = \frac{(z_4 - z_1)(z_2 - z_3)}{(z_4 - z_3)(z_2 - z_1)}$$

which is called the cross ratio of the four points.

(a) Show that the cross ratio makes sense when one of the four points is allowed to take on the value $\infty \in \hat{\mathbb{C}}$. Thus, the cross ratio is defined for any four distinct points in $\hat{\mathbb{C}}$.

(b) Let $\gamma \in PSL_2(\mathbb{C})$. Show that $\chi(\gamma(z_1), \gamma(z_2), \gamma(z_3), \gamma(z_4)) = \chi(z_1, z_2, z_3, z_4)$.

(c) Prove the following alternate characterization of the cross ratio: "There exists a unique Möbius transformation f such that $f(z_1) = 0$, $f(z_2) = 1$, and $f(z_3) = \infty$. Furthermore, $f(z_4) = \chi(z_1, z_2, z_3, z_4)$."

(d) Show that $\chi(z_1, z_2, z_3, z_4)$ is real if and only if z_1, z_2, z_3, z_4 lie on a circle. Moreover, $\mathrm{im}(\chi) > 0$ if and only if z_4 lies inside the unique circle containing z_1, z_2, z_3, where "inside" means that z_4 is always on your left if you walk around the circle in the direction that visits z_1, z_2, z_3 in that order.

(2) (*Schottky details*) Complete problems 3.4, 3.5, 3.6, and 4.5 in *Indra*, thus filling in the details in our discussion of the limit set of a Schottky group.

(3) Write a formula for a Möbius transformation that maps $C(z, r)$ to $C(z', r')$, where as before $C(z, r)$ is the circle with center z and radius r. (*Hint: First map an arbitrary circle to the real line.*

(4) Show that a Möbius transformation γ takes $\mathbb{H} = \{z \in \mathbb{C} \mid \mathrm{im}(z) > 0\}$ into itself (i.e. $\gamma(z) \in \mathbb{H}$ for all $z \in \mathbb{H}$) if and only if γ can be represented by a matrix in $SL_2(R)$. Show that γ maps $\mathbb{R}$ to itself if and only if it can be represented by a matrix with real entries. (Consider $\gamma(z) = -z$ to highlight the difference between these two statements.)

(5) Given four disjoint disks in $\mathbb{C}$, there are many ways to choose two loxodromic Möbius transformations that pair these disks so as to generate a Schottky group. For a fixed choice of four disks, describe the set of *all* pairs of Möbius transformations that pair them as in the definition of a Schottky group. This parameter space should turn out to be a manifold. What is its dimension? Could you predict this dimension using geometric arguments?

(6) (*Fixed points and algebra*) Read note 7.2 in *Indra.*

(a) Let F_i denote the sequence of Fibonacci numbers, $\{F_1, F_2, F_3, \ldots\} = \{1, 1, 2, 3, 5, 8, \ldots\}$, satisfying $F_i + F_{i+1} = F_{i+2}$. Let

$$R = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 2 & 1 \\ 1 & 1 \end{smallmatrix}\right) \in SL_2(\mathbb{Z}).$$

Show that

$$R\begin{pmatrix} F_{i+1} \\ F_i \end{pmatrix} = \begin{pmatrix} F_{i+3} \\ F_{i+2} \end{pmatrix}$$

for any $i \in \mathbb{N}$.

(b) Let $\phi = \frac{1+\sqrt{5}}{2}$, the *Golden Ratio.* Interpreting R as a Möbius transformation, show that it is hyperbolic and $\phi \in \mathbb{R}$ is its attracting fixed point. Conclude that the ratios of consecutive Fibonacci numbers approach ϕ.

(c) To a sequence $(a_0, a_1, a_2, \ldots)$ of positive integers, one can associate the continued fraction

$$\eta = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cdots}}}$$

whose numerical value is meant to be interpreted as the limit of the sequence

$$a_0,\ a_0 + \frac{1}{a_1},\ a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2}},\ \ldots$$

if the limit exists. Show that for the sequence $(1, 1, 1, 1, \ldots)$, $\eta = 1 + \frac{1}{\eta}$. Conclude that $\eta = \phi$.

(d) Notice that

$$\phi = 1 + \frac{1}{\phi} = 1 + \frac{1}{1 + \frac{1}{\phi}} = \frac{2\phi + 1}{\phi + 1}$$

is exactly the equation that implies ϕ is a fixed point of R. Generalize this idea to show that if $\eta(a, b)$ is the value of the continued fraction associated to the periodic sequence $(a, b, a, b, a, b, \ldots)$, then $\eta(a, b)$ is the attracting fixed point of

$$\begin{aligned} R(a, b) &= \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)^a \left(\begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix}\right)^b \\ &= \left(\begin{smallmatrix} 1 & a \\ 0 & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & 0 \\ b & 1 \end{smallmatrix}\right) \end{aligned}$$

(e) More generally, show that the value of the continued fraction for the periodic sequence $(a_0, a_1, \ldots, a_N, a_0, a_1, \ldots, a_N, \ldots)$, where N is even, is a fixed point of the Möbius transformation with matrix

$$R(a_0, \ldots a_N) = \left(\begin{smallmatrix} 1 & a_0 \\ 0 & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & 0 \\ a_1 & 1 \end{smallmatrix}\right) \cdots \left(\begin{smallmatrix} 1 & a_N \\ 0 & 1 \end{smallmatrix}\right)$$

(f) Prove that $SL_2(\mathbb{Z})$ is generated by $\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix}\right)$.

(Therefore, expressing a matrix $R \in SL_2(\mathbb{Z})$ as a product of these generators is like finding a periodic continued fraction expression for the fixed points of R. It is a fact that the set of numbers that can be expressed as *eventually periodic* continued fractions is exactly the set of of roots of quadratic polynomials with integer coefficients.)