Groups of Möbius Transformations

Homework 1

Note: Problems are of widely varying difficulty!

(1) Show that the set of loxodromic elements is open and dense in $PSL_2(\mathbb{C})$. Give explicit examples of sequences of loxodromic Möbius transformations converging to $T(z) = z + 1$ and $R(z) = iz$.

(2) Construct a subgroup of $PSL_2(\mathbb{C})$ isomorphic to $(\mathbb{Z}/2\mathbb{Z}) \times (\mathbb{Z}/2\mathbb{Z})$. Show that any two such subgroups are conjugate.

(3) For any $\gamma \in PSL_2(\mathbb{C})$, let $H_\gamma \subset PSL_2(\mathbb{C})$ be the closure (in the topological sense) of the subgroup generated by γ. Show that H_γ is a subgroup of $PSL_2(\mathbb{C})$. What abstract groups arise as H_γ?

(4) Prove the following fact mentioned in lecture: The image of a circle in $\hat{\mathbb{C}}$ under a Möbius transformation is also a circle. Recall that the intersection of a circle in $\hat{\mathbb{C}}$ with the complex plane $\mathbb{C}$ is either a circle or a straight line. (*Hint: It suffices to prove this for a set of Möbius transformations that generate* $PSL_2(\mathbb{C})$.)

(5) Let $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in PSL_2(\mathbb{C})$, and let $C(z, r) = \{w \in \mathbb{C} \mid |w - z| = r\}$, the circle of radius r with center z. By the result in the previous exercise, $\gamma(C(z, r)) = C(z', r')$ for some z' and r', except in some exceptional cases where the image is a straight line. Compute z' and r' in terms of z, r, a, b, c, d, and explain how the exceptional cases appear as limits of your formula.

(6) (*Computing with Anti-Möbius transformations*) Let $\tilde{G}$ denote the group of maps $\hat{\mathbb{C}} \to \hat{\mathbb{C}}$ generated by $G = PSL_2(\mathbb{C})$ and the complex conjugation map $c(z) = \bar{z}$.
   (a) Show that $\tilde{G}$ consists of maps of the form $f(z) = \frac{az+b}{cz+d}$ or $g(z) = \frac{a\bar{z}+b}{c\bar{z}+d}$, where $ad - bc \neq 0$. One might call a map of the latter sort an *anti-Möbius transformation.*
   (b) Show that for any circle C in $\hat{\mathbb{C}}$, there is a unique anti-Möbius transformation $I_C \in \tilde{G}$ that fixes each point in C and exchanges the inside and outside of C. This I_C is called the *inversion in* C. (Example: $c(z) = \bar{z}$ fixes $\hat{\mathbb{R}} = \mathbb{R} \cup \infty$ and exchanges the upper and lower half-planes.)
   (c) One can represent an anti-Möbius transformation $z \mapsto \frac{a\bar{z}+b}{c\bar{z}+d}$ as a matrix $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$, where $ad - bc = -1$ so as to distinguish it from an ordinary Möbius transformation. In this way, all elements of $\tilde{G}$ can be represented using matrices of complex numbers. Write out the composition rule for two such matrices, which is no longer the ordinary matrix multiplication.
   (d) Write out the inversion in the circle $C(z, r)$ in matrix form. How can one determine whether or not $\gamma \in \tilde{G}$ is the inversion in *some* circle? Assuming that $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ does represent an inversion, determine the center and radius of the fixed circle in terms of a, b, c, d.
   (e) Show that if $\gamma \in G$, and I_C is the inversion in C, then $\gamma I_C \gamma^{-1}$ is the inversion in $\gamma(C)$. Use the results in the previous part to re-derive the formula for the center and radius of the image of $C(z, r)$ under γ.

(7) (*Lattices in* $\mathbb{C}$) As discussed in lecture, the group $\Gamma_\tau = \langle z \mapsto z+1, z \mapsto z+\tau \rangle$ acts on $\mathbb{C}$ to give a quotient torus E_τ for any $\tau \in (\mathbb{C} - \mathbb{R})$. Furthermore, any group generated by two translations is conjugate to Γ_τ for some τ. Since Γ_τ and $\Gamma_{-\tau}$ are the same as subgroups of $PSL_2(\mathbb{C})$, it suffices to consider $\tau \in \mathbb{H} = \{z \in \mathbb{C} \mid \operatorname{im}(z) > 0\}$.

(a) Show that $\Gamma_\tau = \Gamma_{\tau+1}$.

(b) Show that Γ_τ is conjugate to $\Gamma_{-1/\tau}$.

(c) Describe the set of all $\gamma \in PSL_2(\mathbb{C})$ such that $\gamma\Gamma_\tau\gamma^{-1}$ is a group of translations.

(d) Consider the subgroup $SL_2(\mathbb{Z}) = \left\{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL_2(\mathbb{C}) \mid a, b, c, d \in \mathbb{Z} \right\} \subset SL_2(\mathbb{C})$, and its image $PSL_2(\mathbb{Z})$ in $PSL_2(\mathbb{C})$. Show that the Möbius transformations in $PSL_2(Z)$ map $\mathbb{H}$ to itself.

(e) Show that Γ_τ is conjugate to Γ_η if and only if there exists a Möbius transformation $\gamma \in PSL_2(\mathbb{Z})$ such that $\gamma(\tau) = \eta$. (*Hint: Examine the group* $\langle \tau \mapsto \tau+1, \tau \mapsto -1/\tau \rangle$.) In particular, there are infinitely many values of τ for which the groups Γ_τ are conjugate to one another.

(f) It would be nice to classify discrete groups generated by two translations up to conjugacy in $PSL_2(\mathbb{C})$, but clearly the parameter τ for Γ_τ is not a good choice, as it is far from uniquely determined. Let $\Gamma \subset PSL_2(\mathbb{C})$ be a discrete group generated by two translations, $(z \mapsto z + a)$ an element of Γ with the shortest translation distance, and $(z \mapsto z + b)$ an element with the shortest translation distance among those that are not equal to $(z \mapsto z \pm a)$ (the "second shortest"). Let $\eta = \pm(b/a)$, choosing sign so that $\tau \in \mathbb{H}$. Show that Γ is conjugate to Γ_η.

(g) This η is not uniquely determined by Γ because there may be several shortest translations (give an example). Show that there are at most two values of η that could arise for a given group Γ.

(h) Draw a picture of the set F of all $\eta \in \mathbb{H}$ that can arise for different discrete groups Γ generated by two translations. Argue that F is a fundamental domain for $PSL_2(\mathbb{Z})$ acting on $\mathbb{H}$, meaning that if $\gamma \in PSL_2(\mathbb{Z})$, then $\gamma(\operatorname{int} F) \cap F = \emptyset$ and F contains at least one point from each $PSL_2(\mathbb{Z})$ orbit.

(i) Describe the quotient $\mathbb{H}/PSL_2(\mathbb{Z})$.