Math 535: Complex Analysis – Spring 2016 – David Dumas

Midterm Exam Solutions

(1) Is it possible to conformally map the punctured complex plane $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$ to the annulus $A = \{z : 1 < |z| < 2\}$? If it is possible, find such a conformal map. If not, prove that it is impossible.

It is not possible. Suppose $f : \mathbb{C}^* \to A$ is holomorphic. Then f is bounded on $\mathbb{C}^*$ (in fact $|f(z)| < 2$ for all z) and hence the isolated singularity of f at $z = 0$ is removable. The resulting extension is a bounded entire function, which by Liouville's theorem is constant, and not conformal.

(2) Suppose $a, b, c, d \in \mathbb{C}$. Show that the following conditions are equivalent:

(i) There exists a linear fractional transformation S such that Sa, Sb, Sc, Sd are the vertices of a square, listed in counter-clockwise order.

(ii) $\dfrac{(a-c)(b-d)}{(b-c)(a-d)} = 2.$

A short solution: Four-tuples of distinct points in $\hat{\mathbb{C}}$ are related by a linear fractional transformation if and only if they have the same cross ratio. All squares are related by a combination of translation and rotations (both linear fractional), so their CCW vertex tuples all have the same cross ratio. Thus it suffices to show that the cross ratio of the vertices of some square is equal to 2. We find $\chi(1, i, -1, -i) = 2$.

Here is a more detailed answer which proves the assertion about cross ratio classifying $\mathrm{PSL}_2\mathbb{C}$-equivalence classes of four-tuples:

Recall that the cross ratio $\chi(a, b, c, d) = \frac{(a-c)(b-d)}{(b-c)(a-d)}$ is invariant under linear fractional transformations. Also, $\chi(a, b, c, d)$ is the image of a under the unique linear fractional transformation taking b, c, d to $1, 0, \infty$, respectively.

(i) $\Longrightarrow$ (ii): Let e be the center of square $Q = SaSbScSd$ and let $\theta = \arg(Sa - e)$. Then $T(z) = e^{-i\theta}(z - e)$ is a linear fractional transformation mapping Q to the square with vertices $\{1, i, -1, -i\}$. Thus we have

$$\begin{aligned}\chi(a, b, c, d) &= \chi(Sa, Sb, Sc, Sd) = \chi(TSa, TSb, TSc, TSd) \\ &= \chi(1, i, -1, -i) = \frac{(1+1)(i+i)}{(i+1)(1+i)} = 2.\end{aligned}$$

(ii) $\Longrightarrow$ (i): Let U be the unique linear fractional transformation taking $2, 1, 0, \infty$ to $1, i, -1, -i$, respectively; this exists because $\chi(1, i, -1, -i) = 2$, and explicitly we have $U = V^{-1}$ where $V(z) = \chi(z, i, -1, -i)$. Let $W(z) = \chi(z, b, c, d)$. Then UW maps a, b, c, d to $1, i, -1, -i$.

(3) Suppose that f is analytic on a region Ω and that $|f(z)-1|<1$ for all $z\in\Omega$. Show that

$$\oint_\gamma \frac{f'(z)}{f(z)}\,dz = 0$$

for every closed curve $\gamma\subset\Omega$.

Recall that the integral of $g(z)dz$ over every closed curve in a region Ω is zero if and only if g is the derivative of a holomorphic function in Ω. Hence we need only show that there is a holomorphic function F in Ω which satisfies $F'(z)=\frac{f'(z)}{f(z)}$.

The given inequality implies that the image of f is contained in the slit plane $\mathbb{C}\setminus\{z: z\in\mathbb{R}, z\leqslant 0\}$. Hence if log denotes the principal branch of the logarithm, the image of f is contained in the domain of log, and $\log(f(z))$ is a holomorphic function in Ω. Therefore

$$\frac{\partial}{\partial z}\log(f(z)) = \frac{f'(z)}{f(z)},$$

and $F(z)=\log(f(z))$ is the desired function.

(4) Let γ be the circle $|z|=2016$ oriented counter-clockwise. Calculate:

$$\oint_\gamma \frac{e^z-1}{z}\,dz$$

The function $f(z)=e^z-1$ is entire and $f(0)=0$, so by the Cauchy integral formula we have

$$\oint_\gamma \frac{e^z-1}{z}\,dz = \oint_\gamma \frac{f(z)}{z-0}\,dz = 2\pi i\,\mathrm{n}(\gamma,0)f(0)=0.$$

(5) Suppose f is holomorphic and has a simple zero at $z=0$. (That is, $f(0)=0$ and $f'(0)\neq 0$.) Show that there does *not* exist a function g holomorphic in an open disk containing 0 such that $f(z)=g(z)^2$ on their common domain.

Suppose for contradiction that $f(z)=g(z)^2$ for z near 0. Then $g(0)^2=f(0)=0$, so $g(0)=0$. By Taylor's theorem we can write $g(z)=zh(z)$ for a function h holomorphic at 0. But then $f(z)=z^2h(z)^2$ for z near 0 and

$$f'(z)=2zh(z)^2+2z^2h(z)h'(z)$$

implies $f'(0)=0$, a contradiction.