Math 535: Complex Analysis – Spring 2016 – David Dumas

Final Exam Solutions

Note: The solutions given here are in many places more detailed than the minimum requirements for full credit.

(1) Calculate: $\displaystyle\int_0^\infty \frac{x^{\frac{1}{5}}}{x^2+1}dx$

Solution. Let $f(z) = \frac{z^{\frac{1}{5}}}{z^2+1}$ where $z^{\frac{1}{5}}$ is the holomorphic branch defined on $\mathbb{C} \setminus \{z \geqslant 0\}$.

Let $\gamma_{\varepsilon,R}$ denote the closed "keyhole contour" which is the concatenation of the following arcs:

- γ_1 = the oriented line segment from $\frac{1}{R}\exp(i\varepsilon)$ to $R\exp(i\varepsilon)$,
- γ_2 = the counterclockwise arc of $|z| = R$ from $R\exp(i\varepsilon)$ to $R\exp(i(2\pi-\varepsilon))$,
- γ_3 = the oriented line segment from $R\exp(i(2\pi-\varepsilon))$ to $\frac{1}{R}\exp(i(2\pi-\varepsilon))$, and
- γ_4 = the clockwise arc of $|z| = \frac{1}{R}$ from $\frac{1}{R}\exp(i(2\pi-\varepsilon))$ to $\frac{1}{R}\exp(i\varepsilon)$.

Define

$$I = \lim_{R\to\infty}\lim_{\varepsilon\to 0}\int_{\gamma_{\varepsilon,R}} f(z)dz.$$

We claim that $I = (1-\exp(2\pi i/5))\displaystyle\int_0^\infty f(x)dx$. This is because the branch of $z^{\frac{1}{5}}$ we defined above extends continuously when approaching $x \in \mathbb{R}^+$ from above or from below, with respective limits that are $x^{\frac{1}{5}}$ or $\exp(2\pi i/5)x^{\frac{1}{5}}$. Hence

$$\lim_{\varepsilon\to 0}\int_{\gamma_1} f(z)dz = \int_{1/R}^{R} f(x)dx$$

$$\lim_{\varepsilon\to 0}\int_{\gamma_3} f(z)dz = \exp(2\pi i/5)\int_{R}^{1/R} f(x)dx$$

Adding these and taking $R\to\infty$ gives the claimed value for I, so it suffices to show that $\displaystyle\lim_{R\to\infty}\lim_{\varepsilon\to 0}\int_{\gamma_i} f(z)dz = 0$ for $i = 2,4$. We use the standard method to get a bound these integrals based on the length of the contour and the size of the integrand.

Considering γ_2 first, we have

$$\int_{\gamma_2} f(z)dz \leqslant (\sup_{z\in\gamma_2}|f(z)|)(\text{length}(\gamma_2)).$$

If $z \in \gamma_2$ then $|z| = R$ and for large R this implies $|z^2+1| \geqslant \frac{1}{2}R^2$. We therefore find $|f(z)| \leqslant 2R^{\frac{1}{5}}/R^2 = 2R^{-\frac{9}{5}}$. We also have length$(\gamma_2) \leqslant 2\pi R$, giving $\displaystyle\int_{\gamma_i} f(z)dz \leqslant 4\pi R^{-\frac{4}{5}}$ which goes to zero as $R\to\infty$.

Turning to γ_4, we have

$$\int_{\gamma_4} f(z)dz \leqslant (\sup_{z\in\gamma_4}|f(z)|)(\text{length}(\gamma_4))$$

If $z \in \gamma_4$ then $|z| = \frac{1}{R}$ and for large R this implies $|z^2+1| \geqslant \frac{1}{2}$ and $|f(z)| \leqslant 2R^{-\frac{1}{5}}$. Since length$(\gamma_4) \leqslant 2\pi R^{-1}$, we find $\int_{\gamma_i} f(z)dz \leqslant 4\pi R^{-\frac{6}{5}}$ which also goes to zero as $R \to \infty$.

Now we use the Residue Theorem to compute I. The simple poles of the integrand are at $\pm i$, points about which the contour has winding number one for all large R and small ε, and we find

$$I = 2\pi i(\mathrm{Res}_{z=i} f(z) + \mathrm{Res}_{z=-i} f(z)).$$

We compute

$$\mathrm{Res}_{z=i} f(z) = \lim_{z\to i}(z-i)\frac{z^{\frac{1}{5}}}{(z^2+1))} = \frac{i^{\frac{1}{5}}}{2i} = \frac{\exp(\pi i/10)}{2i}$$

noting that i has argument $\pi/2$ for the purposes of computing our chosen branch of $z^{\frac{1}{5}}$. Similarly,

$$\mathrm{Res}_{z=-i} f(z) = \lim_{z\to -i}(z+i)\frac{z^{\frac{1}{5}}}{(z^2+1))} = -\frac{(-i)^{\frac{1}{5}}}{2i} = -\frac{\exp(3\pi i/10)}{2i}$$

Using the above formula for I we have, finally

$$\int_0^\infty \frac{x^{\frac{1}{5}}}{x^2+1}dx = \frac{2\pi i\left(\frac{\exp(\pi i/10)}{2i} - \frac{\exp(3\pi i/10)}{2i}\right)}{1-\exp(2\pi i/5)}$$

Further simplification is not necessary on an exam, but with a bit of algebra this can be reduced to the tidy expression

$$\int_0^\infty \frac{x^{\frac{1}{5}}}{x^2+1}dx = \pi\frac{\sin(\pi/10)}{\sin(\pi/5)}.$$

(2) Does there exist an entire function f with the following properties?

- $f(1) = 0$
- $f(2) = 0$
- $f(z) \in \mathbb{R}$ if and only if $z \in \mathbb{R}$

Either give an example of such a function, or prove that no such function exists.

Solution. No such function exists. In fact, we claim that any holomorphic function which has $f(1) = 0$, $f(2) = 0$, and which is real for all $z \in \mathbb{R}$ is also real at some point in $\mathbb{C} \setminus \mathbb{R}$. If f is identically zero this is immediate, so assume from now on that f is not identically zero.

Since f is real-valued and differentiable on $[1,2]$, by Rolle's theorem there exists $c \in [1,2]$ such that $f'(c) = 0$. Let $g(z) = f(c+z) - f(c)$, so that $g(0) = g'(0) = 0$. Let $k \geqslant 2$ be the order of this zero of the function g. Then the local standard form for the holomorphic map g is

$$g(z) = h(z)^k$$

for some holomorphic function h with $h(0) = 0$ and $h'(0) \neq 0$. In particular h is conformal on some small open disk about 0. The condition $g(z) \in \mathbb{R}$ is equivalent, for z near 0, to $\arg h(z) \in \frac{\pi}{k}\mathbb{Z}$. Since the local inverse of h is a conformal map, we find that near $z = 0$, the set $g^{-1}(\mathbb{R})$ consists $2k$ smooth arcs emanating from 0 whose tangent vectors have arguments differing by all integer multiples of π/k. At most two of these arcs are tangent to $\mathbb{R}$, and $k \geqslant 2$, so one of these arcs contains a non-real point z_0 (which by construction has $g(z_0) \in \mathbb{R}$). Since c and $f(c)$ are both real, we conclude $f(c+z_0) = g(z_0) + f(c)$ is real while $c + z_0$ is not.

Remark. There are lots of different solutions to this problem. Another one is to consider the image by f of a large circle which encloses 1 and 2. By the argument principle, the image has winding number 2 about the origin. However, the given conditions on f would imply that the image crosses $\mathbb{R}$ at only two points (the images of the real points on the circle), which can be used to show that the winding number is zero or one, giving a contradiction.

It is also possible to attack the problem by considering the imaginary part of f, which is a harmonic function vanishing only on $\mathbb{R}$, and using the Cauchy-Riemann equations to infer from this that the real part of f cannot have multiple zeros on $\mathbb{R}$.

(3) For $n \in \mathbb{N}$ let $\Lambda_n \subset \mathbb{C}$ denote the lattice generated by $\omega_1 = 1$ and $\omega_2 = ni$. Let $\wp_n$ denote the Weierstrass function of Λ_n. Identify the limit of the meromorphic functions $\wp_n$ as $n \to \infty$, and the region on which the convergence is locally uniform.

Solution. We will show that the limit is $\pi^2 \csc^2(\pi z) - \frac{\pi^2}{3}$, with locally uniform convergence in $\mathbb{C} \setminus \mathbb{Z}$. First recall that

$$\pi^2 \csc^2(\pi z) = \sum_{n\in\mathbb{Z}} \frac{1}{(z-n)^2}$$

with locally uniform convergence on $\mathbb{C} \setminus \mathbb{Z}$. (It is equivalent to say: "With uniform convergence on every closed disk, once we omit the terms with poles in that disk".) Also recall that

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

Since $(-n)^2 = n$, we can rewrite this as half of the corresponding sum over $\mathbb{Z} \setminus \{0\}$, using $(-n)^2 = n$, obtaining

$$\sum_{n\in(\mathbb{Z}\setminus\{0\})} \frac{1}{n^2} = \frac{\pi^2}{3}.$$

Adding this to the series identity for the cosecant function and renaming n to ω, we find

$$\pi^2 \csc^2(\pi z) - \frac{\pi^2}{3} = \frac{1}{z^2} + \sum_{\omega\in(\mathbb{Z}\setminus\{0\})} \left(\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2} \right).$$

Recalling the definition of the function $\wp$, we see that the sum above consists of the terms from $\wp_n$ that lie on the real axis. To show that this is equal to the limit as $n \to \infty$ of the full sum, we need to show that the sequence of functions defined by the remaining terms, i.e.

$$f_n(z) = \sum_{\omega\in(\Lambda_n\setminus\mathbb{R})} \left(\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2} \right)$$

converges locally uniformly to zero as $n \to \infty$.

To show this, recall that the sum defining $\wp_n$ itself converges in $|z| \leqslant R$ because, after omitting the finitely many terms with $|\omega| \leqslant 2R$, we have

$$\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2} \leqslant \frac{C|z|}{|\omega|^3}$$

for a fixed constant C, hence the remaining terms of $\wp_n$ have as majorant in $|z| \leqslant R$ the absolutely convergent series

$$CR \sum_{\omega \in (\Lambda_n \setminus \{0\})} \frac{1}{|\omega|^3}$$

Applying the same reasoning to f_n, there is no need to omit any terms if n is large enough, since $\omega \in (\Lambda_n \setminus \mathbb{R})$ implies $|\omega| \geqslant n$, so we have

$$|f_n(z)| \leqslant CR \sum_{\omega \in (\Lambda_n \setminus \mathbb{R})} \frac{1}{|\omega|^3}$$

again on $|z| \leqslant R$. Thus our task is reduced to showing that

$$\sum_{\omega \in (\Lambda_n \setminus \mathbb{R})} \frac{1}{|\omega|^3}$$

is small when n is large. Define $\Xi_n = \{(\ell + mi) \ : \ \max(|\ell|, |m|) \geqslant n\}$, so that $\Lambda_n \subset \Xi_n$ and

$$\sum_{\omega \in (\Lambda_n \setminus \mathbb{R})} \frac{1}{|\omega|^3} \leqslant \sum_{\omega \in \Xi_n} \frac{1}{|\omega|^3}.$$

We will estimate the sum on the right, which is easily seen to converge, for example since $\Xi_n \subset \Lambda_1$ and the sum of $|\omega|^{-3}$ over *any* lattice converges.

Consider the terms in $\sum_{\omega \in \Xi_n} \frac{1}{|\omega|^3}$ with $\omega = (\ell + mi)$ and $\max(|\ell|, |m|) = k$. The set of such terms is nonempty for each $k \geqslant n$ in which case it has $8k$ elements, each contributing at most k^{-3} to the sum. We find,

$$\sum_{\omega \in \Xi_n} \frac{1}{|\omega|^3} \leqslant \sum_{k=n}^{\infty} 8k \frac{1}{k^3} = 8 \sum_{k=n}^{\infty} \frac{1}{k^2}$$

which is the tail of a convergent series with the first $(n-1)$ terms omitted. Thus as $n \to \infty$ this remainder goes to zero, as required.

(4) Suppose f is a holomorphic function on $|z| < 2$ that is *even* (that is, $f(-z) = f(z)$). Show that there exists a holomorphic function F on the annulus $1 < |z| < 2$ such that

$$F'(z) = \frac{f(z)}{z^2 - 1}.$$

Solution. Let $A = \{z \ : \ 1 < |z| < 2\}$ and let $g(z) = \frac{f(z)}{z^2-1}$. Such F exists if and only if the integral of g over every closed path in A is equal to zero. Any closed path in A is homologous to an integer multiple of the circle $C = \{|z| = \frac{3}{2}\}$, so we need only show that

$$\int_C g(z) dz = 0.$$

Since g is meromorphic in $|z| < 2$, we can compute the integral above by residues:

$$\int_C g(z)dz = 2\pi i(\mathrm{Res}_{z=1}g(z) + \mathrm{Res}_{z=-1}g(z)).$$

Since $\frac{1}{z^2-1}$ has simple poles at $z = \pm 1$ with residues $\pm\frac{1}{2}$, the residues of g are

$$\mathrm{Res}_{z=1}g(z) = f(1)\mathrm{Res}_{z=1}\frac{1}{z^2-1} = \frac{1}{2}f(1)$$

and

$$\mathrm{Res}_{z=-1}g(z) = f(-1)\mathrm{Res}_{z=-1}\frac{1}{z^2-1} = -\frac{1}{2}f(-1).$$

Since f is even, these residues sum to zero as required.

(5) Completely describe the convergence of the power series

$$\sum_{n=1}^{\infty}\frac{z^{2n}}{2^n n^3}$$

for $z \in \mathbb{C}$. That is, determine the set of all z for which the series converges, and separately, identify the largest open set in which the convergence is locally uniform.

Solution. This radius of convergence R of a power series satisfies

$$R^{-1} = \limsup_{n\to\infty}|a_k|^{1/k}$$

where a_k is the coefficient of z^k. In this case $a_k = 0$ for odd k and $a_{2n} = 2^{-n}n^-3$. Thus

$$R^{-1} = \limsup_{n\to\infty}(2^{-n}n^{-3})^{\frac{1}{2n}}.$$

Note that $(2^{-n})^{\frac{1}{2n}} = 2^{-\frac{1}{2}}$, so it will follow that $R = \sqrt{2}$ if we show

$$\lim_{n\to\infty} n^{\frac{3}{2n}} = 1.$$

Taking the logarithm gives

$$\log\lim_{n\to\infty} n^{\frac{3}{2n}} = \frac{3}{2}\lim_{n\to\infty}\frac{\log n}{n} = 0$$

as required.

We therefore conclude that the series in question converges locally uniformly in the open disk $|z| < \sqrt{2}$ and that it diverges if $|z| > \sqrt{2}$.

It remains only to consider what happens for $|z| = \sqrt{2}$. For such z the series becomes $\sum_n \frac{w^n}{n^3}$ where $w = \frac{1}{2}z^2$ has $|w| = 1$. Thus at such points the series has the convergent series $\sum_n \frac{1}{n^3}$ as a majorant, and in particular converges.

To summarize, the series converges if and only if $|z| \leqslant \sqrt{2}$, and it converges locally uniformly in $|z| < \sqrt{2}$ (and not in any larger open set).

(6) Find all linear fractional transformations T such that $T(1) = 1$, $T(3) = 3$, and $T(T(z)) = z$ for all z.

Solution. In this solution we use multiplicative notation for composition of linear fractional transformations, so e.g. ST refers to the composition $S \circ T$ if $S(z)$ and $T(z)$ are linear fractional. We also use I to denote the identity map, $I(z) = z$.

A linear fractional transformation which fixes 0 and ∞ in $\hat{\mathbb{C}}$ has the form $F(z) = \lambda z$ for some $\lambda \in \mathbb{C}^*$. If such a transformation has $FF = z$ then $\lambda^2 = 1$ and there are two possibilities: $F = I$ or $F(z) = -z$.

Let $S(z) = \frac{z-1}{z-3}$. This linear fractional transformation satisfies $S(1) = 0$ and $S(3) = \infty$, so if T is as described in the problem, then STS^{-1} is linear fractional, fixes 0 and ∞, and has $STS^{-1}STS^{-1} = STTS^{-1} = SIS^{-1} = I$. Thus the possible transformations T are $S^{-1}FS$ where $F = I$ or $F(z) = -z$. The first is simply I, the latter is easily computed to be

$$T(z) = \frac{2z-3}{z-2}.$$

(7) Find all holomorphic functions on $\mathbb{C}^*$ that satisfy:

$$|f(z)| < |z| + |\log|z||$$

Solution. Let us call this inequality (*).

We will show that the functions satisfying (*) are exactly the linear functions $f(z) = az+b$ where $|a| + |b| < 1$.

First, we show that a function satisfying (*) has a removable singularity at the origin, and hence defines an entire function. Consider the function $zf(z)$. Then for $|z| = r < 1$ we have $|zf(z)| \leqslant r^2 + r\log\frac{1}{r}$ which goes to zero as $r \to 0$. Thus $zf(z)$ is bounded near zero, the singularity of $zf(z)$ is removable, and the extended function g vanishes at $z = 0$. But then $f(z) = g(z)/z$ has a removable singularity at 0, giving the desired extension of f.

Next, we show f is linear. An entire function with a pole of order k at infinity is a polynomial of degree k, so it suffices to show that f has at most a simple pole at infinity, or equivalently that $f(1/z)$ has a simple pole at $z = 0$. By (*) we have

$$f(1/z) < \frac{1}{|z|} + |\log\frac{1}{|z|}| = \frac{1}{|z|} + |\log|z||$$

Arguing as above we find $|zf(1/z)|$ is bounded near $z = 0$, hence extends holomorphically, and $f(1/z)$ is expressible as $\frac{1}{z}g(z)$ for g holomorphic near 0. That is, $f(1/z)$ has at most a simple pole at $z = 0$.

Now we must determine which linear functions $az + b$ satisfy (*). For $|z| = 1$, inequality (*) becomes $|f(z)| < 1$. If either of a or b is zero, this shows the other has absolute value less than one. Otherwise, taking $z = \frac{b|a|}{a|b|} \in S^1$ we find $|f(z)| = |a| + |b|$ and so again (*) gives $|a| + |b| < 1$, and we conclude this condition is *necessary*.

Finally, we show $|a| + |b| < 1$ is *sufficient* for $f(z) = az + b$ to satisfy (*). Note that 1 is the absolute minimum value of $r + |\log r|$ on $(0, \infty)$. If $|z| = r \leqslant 1$ then

$$|az + b| \leqslant |a|r + |b| \leqslant |a| + |b| < 1 \leqslant r + |\log r|$$

and so (*) is satisfied for such z. On the other hand, if $r > 1$ then

$$|az + b| \leqslant |a|r + |b| \leqslant (|a| + |b|)r < r < r + |\log r|$$

and (*) is satisfied for these z as well.

(8) Determine whether or not each family of holomorphic functions on the unit disk is normal:

(a) $\mathcal{F}_1 = \{f : \Delta \to \mathbb{C} \ : \ f(z) \neq 0 \text{ for all } z \in \Delta\}$

(b) $\mathcal{F}_2 = \{f : \Delta \to \mathbb{C} \ : \ f(z) \notin [0,1] \text{ for all } z \in \Delta\}$

(c) $\mathcal{F}_3 = \{f : \Delta \to \mathbb{C} \ : \ |f(z)| > 1 \text{ for all } z \in \Delta\}$

Solution.

(a) $\mathcal{F}_1$ is not normal.

Consider the sequence of functions $f_n = (z+1)^n \in \mathcal{F}_1$. Then $f_n(0) = 1$ but for any $x \in (0,1)$ we have $f_n(x) \to \infty$ as $n \to \infty$, so no subsequence of f_n can converge to a function continuous at zero, nor does any subsequence tend to infinity locally uniformly.

(b) $\mathcal{F}_2$ is normal, and

(c) $\mathcal{F}_3$ is normal.

In fact, since $\mathcal{F}_3 \subset \mathcal{F}_2$, and a subfamily of a normal family is normal, it suffices to show that $\mathcal{F}_2$ is normal.

Recall that $h(z) = z + \frac{1}{z}$ is a conformal map from the complement of the unit disk to the complement of $[-2,2]$. Thus $H(z) = \frac{1}{4}h(z) + 1$ is a conformal map from the complement of the unit disk to the complement of $[0,1]$. Note that both H and its inverse have a simple pole at infinity.

If f_n is a sequence in $\mathcal{F}_2$, then $g_n(z) = \frac{1}{H^{-1}(f_n(z))}$ is a sequence of holomorphic functions to Δ^*. As these are uniformly bounded, there exists a locally uniformly convergent subsequence g_{n_k}. By Hurwitz's theorem, the limit function g_∞ is either nowhere zero or identically zero. In the former case we find that

$$f_{n_k}(z) = H\left(\frac{1}{g_{n_k}(z)}\right)$$

converges locally uniformly to $H(\frac{1}{g_\infty})$. In the latter case, f_{n_k} converges locally uniformly to infinity, since $\frac{1}{g_{n_k}} \to \infty$ and H has a pole at infinity. Thus $\mathcal{F}_2$ is normal.