Math 535 – Complex Analysis – David Dumas

Challenge Problems

Last updated April 14, 2016

(C1) Show that any zero of the Riemann zeta function with nonzero imaginary part lies on the line $\mathrm{Re}(z) = \frac{1}{2}$.

(C2) Let $f(z)$ be an analytic function defined on $\mathbb{C}$. The *Newton map* of $f(z)$ is the function

$$N_f(z) = z - \frac{f(z)}{f'(z)}.$$

Newton's method is a computational method for locating zeros of the function $f(z)$, starting from an initial guess z_0. We define a sequence $\{z_i\}$ by $z_{i+1} = N_f(z_i)$. We say that the method *succeeds* if the sequence z_i converges to a point z_∞ such that $f(z_\infty) = 0$, and that it *fails* otherwise. (Note that the method may fail because $N_f(z_k) = \infty$ for some k, in which case the iteration stops.)

(a) If $p(z) = (z-A)(z-B)$ is a quadratic polynomial with distinct roots $A, B \in \mathbb{C}$, show that Newton's method succeeds if and only if z_0 is closer to one of the two roots; equivalently, the method fails if and only if z_0 lies on the line perpendicularly bisecting the segment AB. (It might be instructive to think about the polynomials $z^2 - 1$ and $z^2 + 1$.)

(b) Say something nontrivial about what happens when Newton's method fails for a quadratic polynomial. (e.g. Why does it fail? Though the sequence does not converge to a root, can you describe its long-term behavior?)

(C3) A smooth function $f(\theta)$ on the unit circle can be extended to a harmonic function $F(z)$ on the unit disk using the Poisson integral formula. Given $f(\theta)$, define another function $g(\theta)$ on the circle by

$$g(\theta) = \frac{1}{2\pi}\int_0^{2\pi} f(\phi)\cot\left(\frac{\theta-\phi}{2}\right) d\phi.$$

Note that the integrand has a singularity at $\theta = \phi$, so the integral should be understood as a principal value

$$g(\theta) = \lim_{\varepsilon\to 0^+} \frac{1}{2\pi}\int_{|\phi-\theta|>\varepsilon} f(\phi)\cot\left(\frac{\theta-\phi}{2}\right) d\phi.$$

Let $G(z)$ denote the harmonic extension of $g(\theta)$. Show that $F(z)$ and $G(z)$ are harmonic conjugates, i.e. that $F(z) + iG(z)$ is an analytic function.

(C4) The identity

$$\cos(n\theta) + i\sin(n\theta) = e^{in\theta} = \left(e^{i\theta}\right)^n = (\cos(\theta) + i\sin(\theta))^n$$

shows that for each integer $n \geqslant 0$ there is a polynomial $T_n(x)$ with the property that

$$T_n(\cos(\theta)) = \cos(n\theta).$$

(a) Explain this. (That is, why does the existence of $T_n(x)$ follow from the identity?)

(b) Write out $T_n(x)$ for $n = 1,2,3,4,5$.

(c) Show that $T_n(x)$ is the coefficient of t^n when the function $\frac{1-xt}{1-2xt+t^2}$ is expanded as a Taylor series in t about the point $t = 0$, i.e.

$$\frac{1-xt}{1-2xt+t^2} = \sum_{n=0}^{\infty} T_n(x)t^n.$$

(C5) Show that one can detect vertices of a regular n-gon in $\mathbb{C}$ with finitely many polynomial conditions; that is, for any $n \geqslant 3$, find a set of polynomials $F_1, F_2, \ldots, F_k$ in n variables such that the complex numbers $a_1, a_2, \ldots, a_n$ are the vertices of a regular n-gon if and only if $F_i(a_1, \ldots, a_n) = 0$ for $i = 1, 2, \ldots, k$.

(C6) Construct a sequence of analytic functions $f_n(z)$ on a domain Ω that converge pointwise to a function $f(z)$ that is *not* analytic. (Note: It should be a bit surprising that this is possible.)

(C7) Give an example of an analytic function defined by a power series $f(z) = \sum a_n z^n$ with radius of convergence $R = 1$ such that $\sum a_n = S$ but $\lim_{z \to 1} f(z)$ does not exist. (Compare to Abel's theorem, which says that $f(z) \to S$ as z approaches 1 non-tangentially.)

(C8) Give an example of an analytic function $f(z)$ on the open unit disk Δ that cannot be extended to an analytic function in a neighborhood of any point in $\partial\Delta$. More precisely, show that for any $z_0 \in \partial\Delta$ it is impossible to find an analytic function $g(z)$ defined in a neighborhood U of z_0 such that $g(z) = f(z)$ on $U \cap \Delta$.

(C9) Suppose $A_n \in \mathrm{PSL}_2(\mathbb{C})$ is a sequence of Möbius transformations such that $\|A_n\| \to \infty$ as $n \to \infty$. (Here we use the notation $\| \begin{pmatrix} a & b \\ c & d \end{pmatrix} \| = \sqrt{|a|^2 + |b|^2 + |c|^2 + |d|^2}$.) Show that there exists a subsequence A_{n_i} and two points $x, y \in \hat{\mathbb{C}}$ such that for any closed disk D that does not contain y, the sequence of functions $A_{n_i}(z)$ converges uniformly to the constant function x on D. (For example, if $A_n(z) = nz$, then no subsequence is necessary, and one can take $y = 0$ and $x = \infty$.)

(C10) Fix a point $p \in \mathbb{C}$ and let $\widetilde{\mathscr{O}_p}$ denote the set of pairs (U, f) where U is an open neighborhood of p and f is an analytic function defined on U. We say that (U, f) and (V, g) are *equivalent as germs*† if there is an open set $W \subset U \cap V$ such that $f(z) = g(z)$ for all $z \in W$; in this case we write $(U, f) \sim (V, g)$.

(a) Show that the set of equivalence classes $\mathscr{O}_p = \widetilde{\mathscr{O}_p} / \sim$ forms a ring, i.e. that pointwise addition and multiplication of functions descend to well-defined operations on the set of equivalence classes.

(b) Show that $\mathscr{O}_p$ has a unique maximal ideal, and that this ideal is generated by the equivalence class of the function $f(z) = (z - p)$.

† *An earlier version of this problem contained an error in the definition of the equivalence relation* $\sim$.

(C11) Let $f(z)$ be an analytic function with $f'(0) \neq 0$.

(a) Show that there is a unique Möbius transformation $A(z)$ satisfying

$$A(0) = f(0)$$
$$A'(0) = f'(0)$$
$$A''(0) = f''(0).$$

(b) Let $g(z) = A^{-1}(f(z))$. Show that $g(0) = 0$, $g'(0) = 1$, and $g''(0) = 0$. Calcluate $g'''(0)$.

(C12) Develop a Poisson integral formula for extending a piecewise continuous function on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ to a harmonic function on the domain $\frac{x^2}{a^2} + \frac{y^2}{b^2} < 1$. (Use the parameterization $\gamma(t) = a\cos(t) + b\sin(t)$.)

(C13) Let $C \subset [0, 1]$ denote the standard middle-third Cantor set. Suppose that f is a bounded analytic function on $\Omega \setminus C$, where Ω is a region containing $[0, 1]$. Show that f extends to an analytic function on Ω.

(C14) Suppose $A(x)$ is a polynomial. Show that the power series

$$f(z) = \sum_{n \geqslant 0} A(n) z^n$$

converges on $|z| < 1$ to a *rational function* of z.

(C15) Let f be an analytic function on a region Ω with $f'(z) \neq 0$ for all $z \in \Omega$. The *Schwarzian derivative* of f is the function

$$S_f(z) = \frac{f'''(z)}{f'(z)} - \frac{3}{2}\left(\frac{f''(z)}{f'(z)}\right)^2.$$

(a) The *nonlinearity* of f is the function $N_f(z) = \frac{f''(z)}{f'(z)}$. Show that $S_f(z) = {N_f}'(z) - \frac{1}{2}N_f(z)^2$.
(b) Show that $S_f(z) \equiv 0$ if and only if f is the restriction of a Möbius transformation to Ω.
(c) Suppose A is a Möbius transformation. Show that $S_{A\circ f}(z) = S_f(z)$.

(C16) Give an example (with proof) of a power series $\sum_{n=0}^{\infty} a_n z^n$ that converges for all $|z| \leqslant 1$, but where the resulting function on the closed unit disk is not continuous.

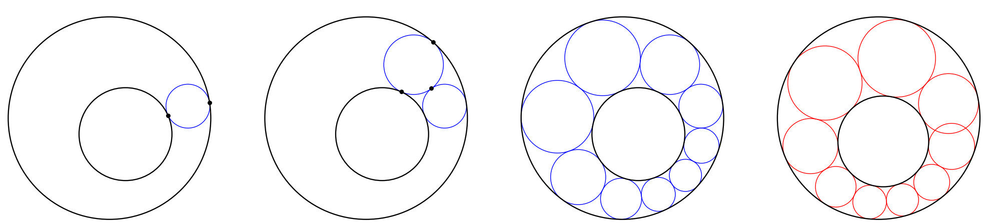
(C17) Let Γ be a group and $\rho : \Gamma \to \mathrm{PSL}_2\mathbb{C}$ is a homomorphism. We say that ρ is *discrete* if the identity element of $\mathrm{PSL}_2\mathbb{C}$ is an isolated point of the set $\rho(\Gamma)$.

Show that if $\rho : \mathbb{Z}\times\mathbb{Z} \to \mathrm{PSL}_2\mathbb{C}$ is a discrete homomorphism, then the image $\rho(\mathbb{Z}\times\mathbb{Z})$ is either a cyclic group (finite or infinite), or else ρ is injective and all of the nontrivial elements of the image are parabolic.

(An equivalent definition of discrete homomorphism that you may use is as follows‡: For any sequence γ_n such that $\rho(\gamma_n)$ converges to the identity element of $\mathrm{PSL}_2\mathbb{C}$, we have that $\rho(\gamma_n)$ is equal to the identity element of $\mathrm{PSL}_2\mathbb{C}$ for all $n > N$.)

‡ *A mistake in this equivalent definition was corrected on April 14, 2016.*

(C18) Suppose two nested circles are given, C and C', as well as a point p on the inner circle C. As shown below, one can form a chain of circles tangent to and lying between C and C', starting with a circle tangent to C at p. This chain may close up neatly, with its last circle tangent to the first (*success*), however it may also end with an overlap instead of a tangency (*failure*). Examples of the construction of the chain and of both possible outcomes are shown below.



(a) Show that success or failure depends only on the pair C, C', and not on the starting point p. (Thus there is a closed chain between them if and only if there are infinitely many distinct chains between them.)
(b) Given centers and radii of C and C', describe how one can determine whether the circle chain construction will succeed, and in the case of success how many circles will be in the chain.

(C19) Consider the claim:

Given three circles $C_1, C_2, C_3 \subset \hat{\mathbb{C}}$, there exists a unique circle D that intersects C_i orthogonally for $i = 1, 2, 3$.

This is not quite true, but it is true for "most" triples of circles. Make this precise, prove it, and characterize the set of orthogonal circles in all other cases. (That is, when is there no such D, and when are there many? When D is not unique, describe all possibilities.)

(C20) Compute $\displaystyle\sum_{n=1}^{\infty}\left(\frac{e^{2\pi n}+1}{e^{2\pi n}-1}\right)\frac{1}{n^7}$.

(C21) (a) Let $A=\{z\,|\,z^n=n \text{ for some } n\in\mathbb{N}\}$. Construct a meromorphic function on $\mathbb{C}$ that has a simple pole at each point of A, and which has any given collection of complex numbers of modulus 1 as residues. (That is, given any function $Q:A\to S^1$, construct a meromorphic function f on $\mathbb{C}$ with $\text{ord}_a f=1$ and $\text{res}_a f=Q(a)$ for all $a\in A$.)

(b) Construct an entire function that has a zero of order 2^n at $\log(n)$ for all $n\in\mathbb{N}$.

(C22) Suppose $E\subset[0,1]$ is a set with positive Lebesgue measure. Show that there exists a bounded non-constant holomorphic function on $\mathbb{C}\setminus E$.

(Note: Such a function cannot be extended holomorphically over E, since by Liouville's theorem it would then be constant. It is natural to compare this problem to C13 where it is shown that bounded holomorphic functions extend over the middle-third Cantor set C. Since the Lebesgue measure of C is zero, this is perfectly consistent.)